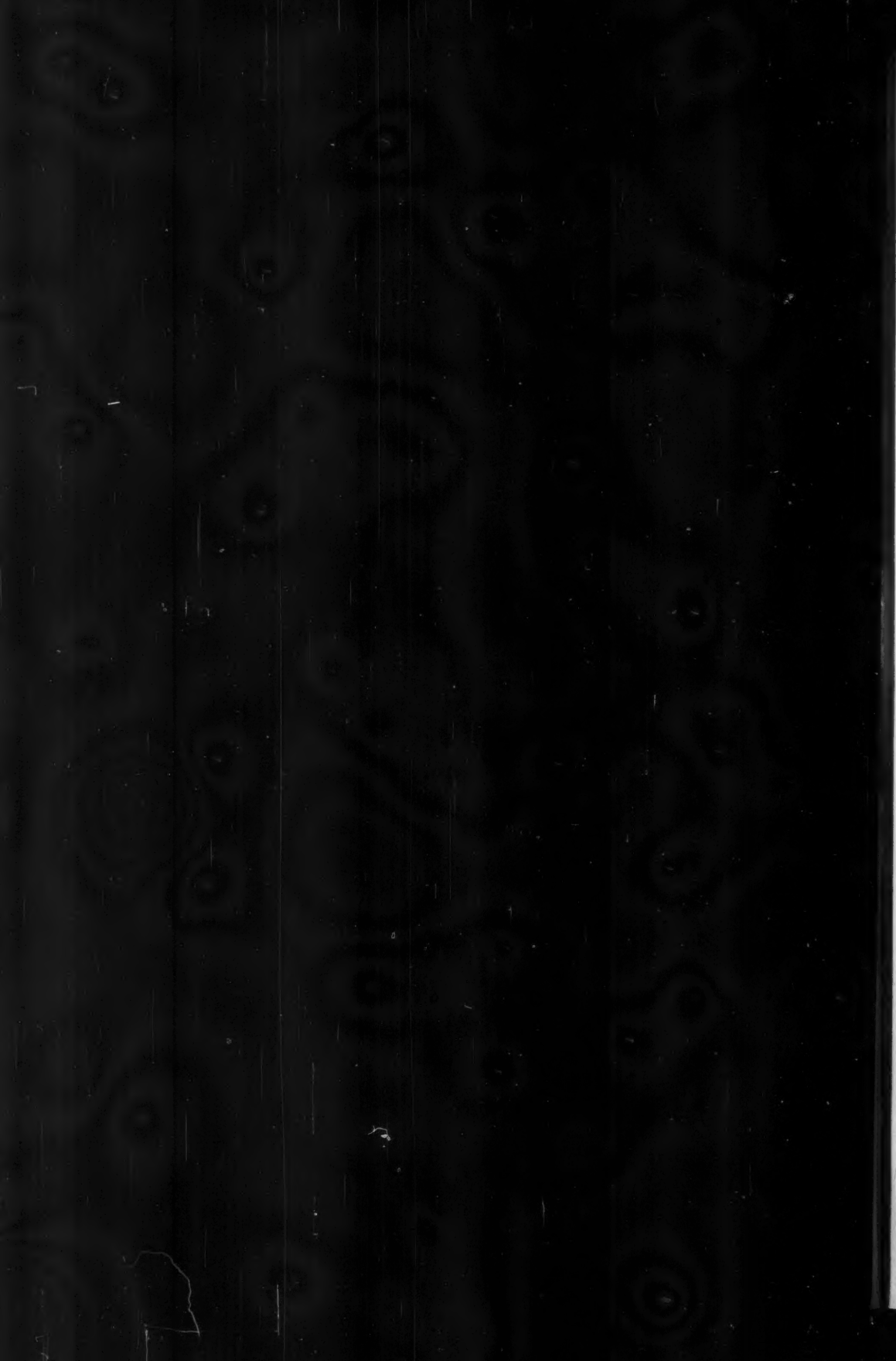

BROOK TAYLOR AND THE MATHEMATICAL THEORY OF LINEAR PERSPECTIVE*

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One can distinguish four overlapping and interrelated periods in the development of the mathematical theory of linear perspective: (1) the "prehistory" period in which, for example, the Greeks are reported to have made some use of perspective drawing in their theater, (2) the 15th and 16th century period of the origin of the theory with the artists-architects-engineers of the Renaissance (Brunelleschi, Franceschi, Alberti, and da Vinici), (3) a period of geometrical expositions typified by the works of del Monte and Stevin in the 17th century, and, finally, (4) the period of a generalized, complete, and even abstract theory.

This last period falls largely in the 18th century and is typified by the work of William Jacob Gravesand in Holland, Humphrey Ditton and Brook Taylor in England, and of the Alsacian (he was born in Mülhausen in the period when it was allied with Switzerland) mathematician Johanne Heinrich Lambert.

Of these, the work of Brook Taylor was certainly the most widely translated and reproduced, although the later work of Lambert rivals it in interest and perhaps in its total effect [1].

Brook Taylor published only two books in his lifetime of 46 years. Both of these appeared in 1715 when he was 30, and both of them exerted wide influence. He is, of course, best known for his *Methodus Incrementorum Directa et Inversa* in which appears the well known expansion of $f(x+h)$ which bears his name.

The other book was *LINEAR / PERSPECTIVE OR, A / New METHOD / Of Representing justly all manner of / OBJECTS as they appear to the EYE / IN ALL / SITUATIONS. / A Work necessary for* PAINTERS, */* ARCHITECTS, *& c. to Judge of, and / Regulate Designs by.* This work is today only rarely and sparingly referred to in histories of either mathematics or art. This alone is of interest in view of a study which shows that the original appeared in four editions (or five, if Ware's revision be counted), the latest as recent as 1811, that it appeared in three translations, one French and two Italian, and that Taylor's English disciples in perspective number nine and were responsible for twelve books and twenty-two editions from 1715 through 1888 [2]. By disciples I here mean men who used Taylor's name in the titles or body of their own works which works in turn followed more or less closely Taylor's sequence and method.

One reason for this lack of recognition of Taylor's *Perspective* is perhaps the same defect as that upon which John Bernoulli is said to have seized when, according to Taylor's grandson and biographer, he called the book "abstruse to all and unintelligible to artists for whom it was more especially written" [3]. I have not found these exact words but it is quite likely both that Bernoulli

* Presented on September 1, 1950 to Section VII of the International Congress of Mathematicians held at Cambridge, Massachusetts.

said them and that one must discount them a little because of the heated and sharp nature of the controversy carried on by these two men in the pages of *Acta Eruditorum* and the *Philosophical Transactions*, beginning with a letter by Bernoulli in 1716 [4]. This controversy was over priority and the proper recognition of sources used in both Taylor's *Methodus Incrementorum* and also in his publications on the vibrating string and on an isoperimetric problem. More than this, however, it was a part of the continuation by their partisans of the Newton-Leibniz Controversy which was not always conducted in a fair and rational vein.

Taylor himself, however, recognized the excessive conciseness and abstractness of his first book on perspective when he expanded it from 42 pages to the 70 found in the second, or 1719 edition, and when he added a few plates showing the application of his method to actual drawings of physical objects in addition to the purely geometric diagrams of the first edition.

A later evaluation by Monge and Lacroix is interesting. In An 9 (1801) they recommended to the Académie des Sciences of the Institut de France that it not sanction the publication of a French translation of the first edition by B. Lavite [5]. In the introduction to their report, however, they remarked that Taylor's work was "distinguished from a crowd of others dealing with perspective by its originality and the fruitfulness of the principle upon which it was based." They also termed it "elegant," "expeditious," and "not lacking in a sort of generality." They explained that they did not favor printing it in spite of this for two reasons; namely, that additional work or study of perspective was unnecessary for those who already knew "Stéréotomie," and that Taylor's work was too geometrical for most artists who were not versed in Stéréotomie. This seems a fair and rational evaluation when one recalls that it was made by the founder of descriptive geometry and one of his followers.

More recently Julian Coolidge has referred to Taylor's work as the "capstone of the whole edifice" of perspective [6]. In spite of this and the fact that Gino Loria also has paid some attention to Taylor's work [7], the tabulation of editions, translations and extensions which is noted above and detailed in the notes has not been made before, nor is there a discussion of the first or 1715 edition available since later writers on perspective used the second edition and the historians have used either it or versions still more remote from the original edition.

In this paper are presented only three of the items of especial interest which appeared in the 1715 edition but not in later editions. First, however, it will be helpful to note that Taylor found it necessary to, as he said, "Consider this subject entirely anew." To this end he gave new terms, four axioms (in the 1719 edition), and then developed his theory in a formal and rigorous fashion with theorems, corollaries, problems, and proofs. He defined the "vanishing line" of any "original plane" to be the intersection with the picture plane of a plane through the eye of the beholder parallel to the original plane. This means that his basic three dimensional diagram as shown in Figure 1 (Plate 1 of Taylor's

book) consisted of four planes parallel in pairs, the picture plane, the "directing plane" through the eye of the beholder and parallel to the picture, the original plane, and the plane through the eye parallel to it. The "vanishing point" of any "original line" is the intersection with the picture of a line through the eye

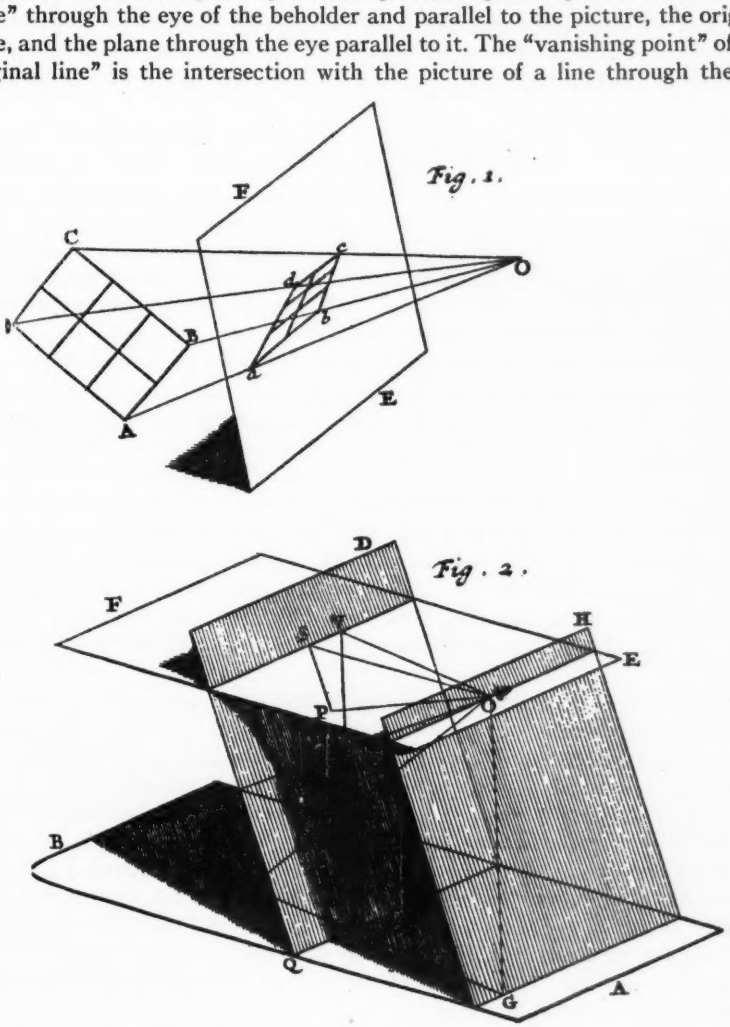


FIG. 1

parallel to the original line. Since the intersection of any original line with the picture is its own perspective, it follows as "PROPOSITION I, THEOREM 1" that *"The representation of a Line is Part of a Line passing thro' the Intersection and Vanishing Point of the Original Line."*

The above discussion of Taylor's terminology and Theorem I indicates three

things about his work; namely, his formal, mathematical formulation, the generality of his concepts and procedures (he has, for example, no need to distinguish a special ground line and horizon line), and the completely new concise synthesis which he did achieve of procedures not all of which were original with him.

The first of the three specific items which will be discussed here is his construction for the perspective of a triangle ABC (see Figure 20 of Taylor's Plate 7 as reproduced in our Figure 2). Consider the plane of the drawing to represent the picture plane with two other planes rotated into coincidence with it. Below ED, the intersection of the original plane and the picture, is the original plane itself, containing ABC (to be thought of as "behind" the picture from the viewpoint of the observer) rotated about ED into the picture. The plane through the eye parallel to the ground plane has been rotated upward about FH. Above FH then is O, the eye point. Extend AB to meet ED in D, its "intersection." Draw a line through O parallel to AB to meet FH in F, the vanishing point of AB. FD is then the indefinite perspective of AB, *i.e.*, the perspective of AB produced. Join O to A and B. The intersections of these lines with FD determine perspective points a and b. A similar determination of c (c could also be located as the intersection of EH and IG) would give a diagram in which the corresponding sides of the two triangles meet on ED and the lines joining corresponding vertices concur in O. Although the Desargues triangle theorem is neither mentioned nor stated, note how completely it is implicit in this construction and the accompanying diagram [8]. Both the problem and the diagram were modified in the second edition and the relationship, though still implicit, became less obvious.

Also in the 1715 edition but omitted in the second edition is the problem of finding the perspective of the shadow of a triangle on a plane. Not only does this associate with the three dimensional case of the Desargues theorem, but of particular interest is Taylor's second solution of the problem which is, as he terms it, by putting the rules of perspective in perspective. In this same vein he elsewhere gives constructions for such things as the vanishing point of lines perpendicular to a given plane for the specific purpose of making it possible to draw directly in perspective without first having an orthogonal projection. In this Taylor anticipated Lambert who took this as one of the major objectives of his *Freye Perspective* (1759). Taylor's work with such problems led him to make repeated use in the 1715 edition of the idea of associating infinitely distant intersections with parallel lines.

A second construction which is both unique to the 1715 edition and which has for its purpose the construction of drawings directly in perspective is Taylor's solution of the problem of completing the construction of the perspective of a circle, given the perspective of its center and of one of its points. The diagram for this is to be found in "Fig. 21" of Taylor's Plate 7 which is our Figure 2. C is the perspective of the center of a circle, A the perspective of a point on

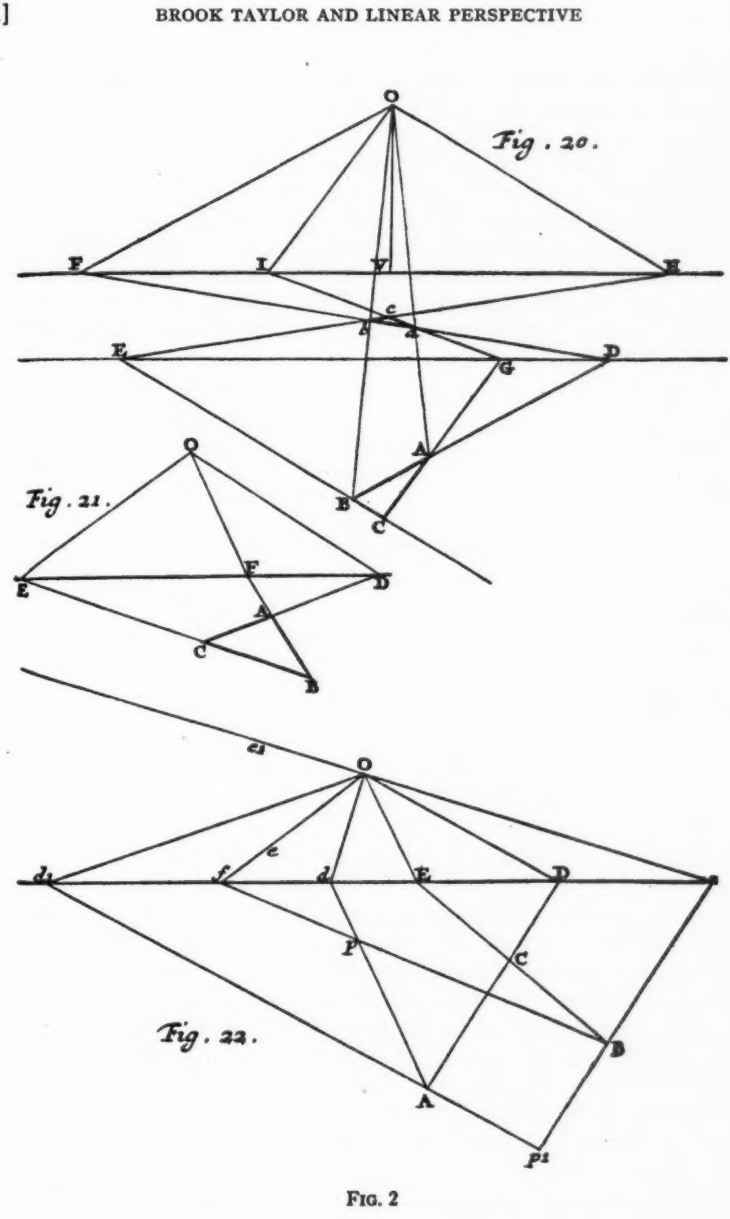


FIG. 2

the circle, *ED* the vanishing line of its plane, and *O* the eye rotated into the picture. *CA* then represents a radius. Draw any line through *C* to meet the vanishing line in *E* and extend *CA* to intersect it in *D*. Bisect angle *EOD* to locate point *F* on *ED*. The join of *F* and *A* meets *EC* in *B*, another point of the perspective circle. Taylor's reasoning was based on the fact that since the angles at *A* and *B* are perspectives of equal angles then *CA* and *CB* are perspectives of the sides of an isosceles triangle and hence are the perspectives of equal lines. *CB* must then represent a radius, and *B* is the perspective of a point on the circle. This is another example of Taylor's thinking and drawing directly in perspective. It is also interesting to note that if the construction were extended

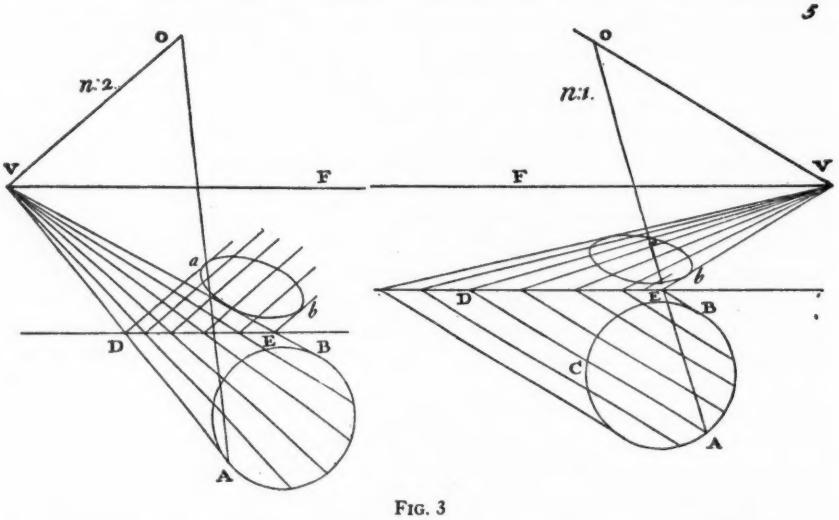


FIG. 3

to determine the second point on each radius by bisecting the supplement of *EOD* we would have an harmonic set of points on *ED*, and further that *C* and *ED* are pole and polar with respect to the conic which represents the circle.

The first explicit use which the author has found of the terms pole and polar in perspective is in the work of Cousinery in 1828. However, John Hamilton, one of Taylor's followers, had also read LaHire's *Sectiones Conicae* (1685) in which much use is made of harmonic sets. Book III of Hamilton's *Stereography or a Compleat Body of Perspective* (1738) makes extensive use of harmonic sets and some use of theorems on poles and polars although without using the latter terms.

Our Figure 3 shows, for contrast with the above, the two constructions for the perspective of a circle which were given as Figure 13 in the 1719 edition of Taylor's book. They are more conventional, use the orthogonal projection of the original circle, and are described in much more detail in the text.

Taylor gives no proof or explanation of the third unique construction which is here presented from the 1715 edition. The construction we refer to is "n:2." in Figure 32 of Taylor's Plate 12 which is shown here as Figure 4. Both "n:1." and "n:2." are constructions for a line through a given point and the inaccessible intersection of two other given lines. Today, "n:2." would be regarded as an application of harmonic sets related to complete quadrilaterals. Knowing that he did use both the idea that lines meeting on a vanishing line are parallel and its converse, we can guess that Taylor might have proven it quickly and

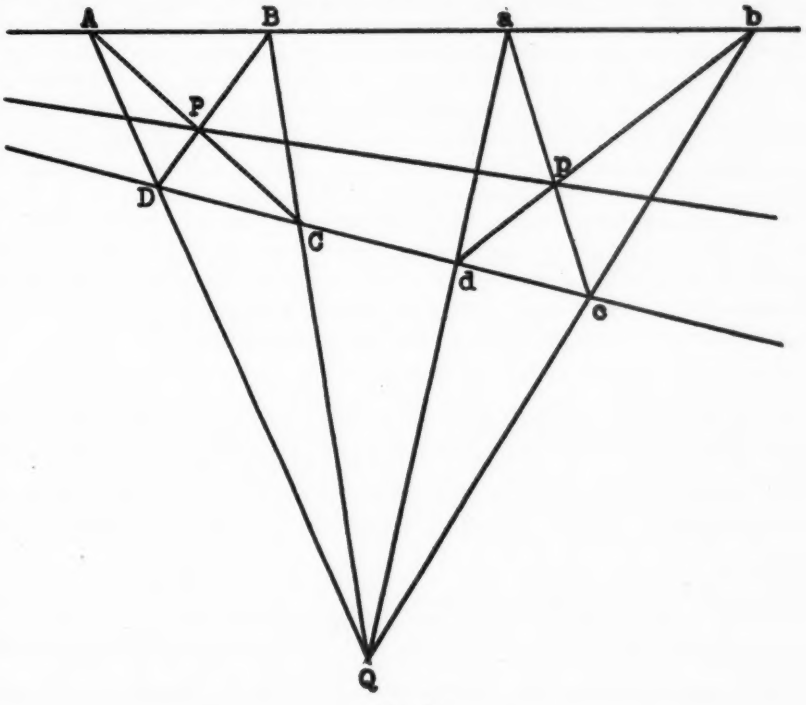


FIG. 4

easily by thinking in a "perspective" geometry where *ABCD* and *abcd* would be parallelograms rather than in a "Euclidean" geometry. In any case, Taylor was the first writer on perspective to treat this problem.

The only original work on perspective printed in England prior to Taylor's was Humphrey Ditton's *A Treatise of Perspective* of 1712, which deserves more note than it has had in the past but which is not comparable to Taylor's *Linear Perspective* in generality or originality. Following Taylor in England only Hamilton showed much originality while on the continent Lambert's work was outstanding in this century. Another feature which, though first met in Guido

Ubaldo Monte's *Perspectivae Librix Sex* (Pisa, 1600), was developed by Taylor and then carried much farther by Lambert was the solution of the inverse problem of perspective. This problem; namely, given data about a perspective drawing to draw inferences about the original, is basic in the modern science of photogrammetry.

This discussion represents only a portion of a complete study which the author has made of the development of the mathematical theory of perspective. It shows that Brook Taylor contributed a mathematically clear, concise, and logical, but abstract, formulation of extraordinary generality, including some treatment of the inverse problem. The editions, translations, and sequels to his work noted here extended his influence beyond both his homeland and his chronological period.

In conclusion, we note for those who might wonder at the interest of Taylor in this subject that not only is this interest consistent with the mathematical and cultural interests of the time (Desargues, Stevin, Ozanam, were earlier mathematicians who wrote on this topic), but also that Taylor grew up in a home where music and art were popular diversions. According to his grandson, Taylor himself in his painting "favored landscapes and water colors.—They have a force of color, a freedom of touch, a varied disposition of planes of distance, and a learned use of aerial as well as linear perspective which all professional men who have seen these paintings have admired" [9].

Notes

1. Max Steck, Johann Heinrich Lambert Schriften zur Perspektive. (Berlin: 1943), p. 48 lists Jacquier's French translation of Taylor's work, as among the books in Lambert's library and adds parenthetically that it was "von Lambert im II Teil des Hauptwerkes benützt."

2. Since the writer found no other at all complete enumeration of these works, it seems appropriate to preserve this data in detail for future reference. The books referred to are:

Editions:

Brook Taylor, Linear Perspective. London: 1715.

———, New Principles of Linear Perspective. London: 1719.

———, New Principles of Linear Perspective—third edition corrected by J. Colson. London: 1749.

———, Method of Perspective. 1766. The Dictionary of National Biography (London: 1899 LIX, p. 359) lists this under Isaac Ware who, it says, prepared the edition.

———, New Principles of Linear Perspective:—The fourth edition, revised. London: 1811.

Translations:

Francois Jacquier, Elementi di perspectiva secondo li principii di Brook Taylor, con varie aggiente. Roma: 1753.

Antoine Rivoire, Nouveaux principes de la perspective lineaire, traduction de deux ouvrages, l'un Anglois, due Docteur Brook Taylor, l'autre Latin, de M. Patrice Murdoch. Amsterdam: 1759.

Jacopo Stellini, Opere varie. Padova: 1781. Contains in volume II Taylor's "Nuovi principij della prospettiva lineare" according to Pietro Riccardi in his Biblioteca Matematica Italiana.

Disciples:

John Hamilton, Stereography or a compleat body of perspective. London: 1738, 1740, 1748.

John Joshua Kirby, Dr. Brook Taylor's method of perspective made easy both in theory and practice. Ipswich: 1754, 1755; London: 1765, 1768.

John Joshua Kirby, The perspective of architecture—deduced from the principles of Dr. Brook Taylor. London: 1761.

———, Dr. Brook Taylor's method of perspective, compared with examples lately published on this subject, as Sirigatti's by Isaac Ware. London: 1767.

Daniel Fournier, A treatise of the theory and practice of perspective. Wherein the principles—laid down by Dr. Brook Taylor are explained by moveable schemes. London: 1761, 1762, 1763, 1764.

Joseph Highmore, Practice of perspective on the principles of Dr. Brook Taylor. London: 1763.

Thomas Malton, A compleat treatise on perspective in theory and practice on the true principles of Dr. Brook Taylor. London: 1775, 1776, 1779.

———, An appendix or second part to the compleat treatise on perspective containing a brief history of perspective. London: 1783.

James Malton, The young painter's maulstick; being a practical treatise on perspective;—with the theoretic principles of—B. Taylor. London: 1800.

Edward Edwards, A practical treatise of perspective on the principles of Dr. Brook Taylor. London: 1803.

Joseph Jopling, Taylor's principles of linear perspective, new edition with additions by Joseph Jopling. London: 1835.

George Blacker, John Heywood's second grade perspective—adapted from Dr. Brook Taylor. Manchester: 1885–88.

3. Contemplatio Philosophica: A Posthumous Work of the Late Brook Taylor, L.L.D., F.R.S. some time Secretary of the Royal Society. To which is prefixed a life of the author by his grandson, Sir William Young, Bart. F.R.S., A.S.S. (London: Printed by W. Bulmer and Co., 1793), p. 29. The title page of this book bears the printed note *Not Published*. The book also includes some letters to and from Taylor to which we will refer later.

4. "Epistola Pro Eminente Mathematico Dn. Johanne Bernoullio, contra quendam ex Anglia antagonistam scripta." Acta Eruditorum. (July, 1716), pp. 296–315. The article preceding this one in Acta was "Methodus Incrementorum Directa & Inversa; Autore Broock (sic!) Taylor, L.L.D. & Regiae Societatis Secretario," a summary of the book with comments, references to Leibniz and his procedures and to Collins' Commercium Epistolicum. This "review" was probably written by Leibniz himself according to Heinrich Auchter, Brook Taylor der Mathematiker und Philosoph. (Wurzburg: Konrad Triltsch, 1937), p. 79.

The Taylor-Bernoulli dispute as it appeared in Acta and the Philosophical Transactions is somewhat expanded in details and clarified by the letters printed by Young in the Contemplatio Philosophica and in Auchter, *op. cit.*

Taylor wrote on February 5, 1719 to Count Raymond de Montmort in reply to a letter from Bernoulli which Montmort had forwarded, "For if the book be so very obscure, as he says it is, that the best artists, those already acquainted with the subjects, cannot well understand it—." This may be the source for Young's quotation. Taylor, however, seems to have been referring to his Methodus rather than his work on perspective.

5. Institut de France, Académie des Sciences, Procès-Verbaux des Séances de l'Académie. Tome II, An VIII–XI (1800–1804), 1912, p. 360 ff.

6. Julian L. Coolige, A History of Geometrical Methods. (Oxford, 1940), p. 108.

7. Gino Loria, Storia della Geometria Descrittiva. (Milano, 1921), pp. 43–51.

8. The copy of Taylor's book used originally in this study is in the Rare Book Room of the University of Michigan. The author, happening recently to have purchased a copy for himself, was startled to find lines *OaA* and *ObB* in his copy to have been drawn in with ordinary pen and

ink after printing. Further comparison of the two copies showed that a number of corrections to the plates were made during the printing process, appearing inked in in the author's copy and printed in the library's copy. It should be remarked that practically all of the original works cited are to be found in the University of Michigan's collection built up by Professor L. C. Karpinski whose suggestions and advice aided significantly in the study of which this paper is a partial report.

9. Sir W. Young, *op. cit.*, pp. 28–29.

THE PROBLEM OF n LIARS AND MARKOV CHAINS

WILLIAM FELLER, Princeton University

1. Introduction. In the Advanced Problems section of this MONTHLY [4288, Vol. 57 (1950) pp. 43–45] we find a solution of the following problem, first treated by A. S. Eddington: If A, B, C, D each speak the truth once in three times (independently), and *A affirms that B denies that C declares that D is a liar, what is the probability that D was telling the truth?* There are only eight different statements which A can make (or deny) and their enumeration shows that the required probability is 13/41.

John Riordan's remark that this problem applies also to switching circuits led me to look for the appropriate probability model. It turns out that the problem of n liars leads directly to the simplest Markov chain, and that natural variations of the same problem correspond to more general chains. The example may be of some didactic use since the theory of Markov chains grows in importance, but simple illustrative examples are hard to get.

2. Model for the simplest Markov chain. In the original formulation, persons C, B, and A issue successively three statements which may be true or false and which may contradict each other. However, taken at face value, each statement would imply either that D is telling the truth or that he lies. Thus, if B denies that C declares that D is a liar, he implies that D tells the truth. If A (rightly or wrongly) denies that B denied *etc.*, then he implies that D is a liar. In a continued process, an even number of denials cancel, and the implication of a statement like "A_1 asserts that A_2 denies that $A_3 \cdots$" depends on the evenness of the total number of denials.

To use a neutral terminology, we shall speak of a chance process with two possible states; at any time the observed state is 1 or 2 according as the last statement implies that D is honest or a liar. We imagine that the statements are issued at times 1, 2, 3, $\cdots$. Initially, or at time 0, the "observed" state is 1 or 2 according as D tells the truth or lies. (In the original problem actually only D and C know the initial state.) Every possible sample sequence of the process is represented by a succession of the digits 1 and 2, and *vice versa*.

The fundamental assumption (to which the Markov character of the process is due) is that each person knows only the statement of the last speaker, but

not the past history of the system. Moreover, at time n the observed state changes or remains unchanged according as the nth speaker tells the truth or lies. Thus we arrive at the following model which represents the simplest Markov chain:

We have a process with two possible states 1, 2. *Initially (or at time* 0*) the probabilities of the two states are* α *and* β, *respectively* $(\alpha+\beta=1)$. *Whatever the development up to time* n, *there is probability* p *that at time* n *the observed state does not undergo a change and probability* $q=1-p$ *that it does. We seek the conditional probabilities* x_n *and* y_n *that the process actually started from state* 1, *given that at time* n *the observed state is* 1 *or* 2, *respectively.*

In the original formulation $\alpha=p=\frac{1}{3}$, $n=3$, and only x_n is required.

3. Each individual step consists in one of the four possible transitions $1\to1$, $1\to2$, $2\to1$, $2\to2$, and the corresponding transition probabilities are, by assumption,

$$p_{11}=p_{22}=p, \qquad p_{12}=p_{21}=q. \tag{1}$$

Suppose now that at a certain time the system is in state j and let $p_{jk}^{(n)}$ be the probability (on this hypothesis) that n steps later the observed state is k. The $p_{jk}^{(n)}$ are called the n-step transition probabilities. Clearly

$$p_{jk}^{(1)}=p_{jk},$$

and

$$p_{jk}^{(2)}=p_{j1}p_{1k}+p_{j2}p_{2k}. \tag{2}$$

More generally, we can calculate $p_{jk}^{(n)}$ from the obvious recursion formulas

$$p_{jk}^{(n+1)}=p_{j1}^{(n)}p_{1k}+p_{j2}^{(n)}p_{2k}. \tag{3}$$

Now these are just the formulas for matrix multiplication: if the matrix (p_{jk}) is denoted by P, then the $p_{jk}^{(n)}$ are elements of P^n.

Since the initial probabilities of the states 1 and 2 are α and β, the probability $a_k^{(n)}$ of observing at time n the state k is obviously

$$a_k^{(n)}=\alpha p_{1k}^{(n)}+\beta p_{2k}^{(n)}. \tag{4}$$

We find therefore for the two required probabilities

$$x_n=\frac{\alpha p_{11}^{(n)}}{a_1^{(n)}} \quad \text{and} \quad y_n=\frac{\alpha p_{12}^{(n)}}{a_2^{(n)}}. \tag{5}$$

For explicit formulas we must, of course, calculate $p_{jk}^{(n)}$. The formula

$$P^n=\frac{1}{2}\begin{pmatrix}1&1\\1&1\end{pmatrix}+\frac{1}{2}(p-q)^n\begin{pmatrix}1&-1\\-1&1\end{pmatrix} \tag{6}$$

follows directly from the canonical decomposition of matrices, and can easily be verified. Substituting from (6) and (4) into (5) we find

$$x_n = \frac{\alpha\{1 + (p-q)^n\}}{1 + (\alpha-\beta)(p-q)^n}, \qquad y_n = \frac{\alpha\{1 - (p-q)^n\}}{1 - (\alpha-\beta)(p-q)^n}. \tag{7}$$

For $n=3$, $\alpha=p=\frac{1}{3}=13/41$ in agreement with the solutions quoted. Moreover, $y_n=7/20$. Clearly $x_n\to\alpha$ as $n\to\infty$. It is interesting that x_n decreases monotonically if $p>q$, while x_n is alternatingly larger and smaller than α if $p<q$.

4. Preferential lying. Up to now we have assumed that the chances of a person telling the truth are in no way dependent on the statement he is supposed to relay. Suppose now that each person has a preference to claim that D is honest. Then a transition $1\to1$ is more probable than $2\to2$, while $1\to2$ is less probable than $2\to1$.

We can treat this general case in the same way as before, provided we replace (1) by the general matrix of transition probabilities

$$p_{11} = p, \qquad p_{12} = q, \qquad p_{21} = q', \qquad p_{22} = p', \tag{8}$$

where, of course, $p+q=p'+q'=1$. All formulas of the preceding section apply, except that (6) must be replaced by

$$P^n = \frac{1}{q+q}\begin{pmatrix} q' & q \\ q' & q \end{pmatrix} + \frac{(p'-q)^n}{q+q'}\begin{pmatrix} q & -q \\ -q' & q' \end{pmatrix}. \tag{9}$$

The final result is now

$$x_n = \frac{\alpha\{q' + q(p'-q)^n\}}{q' + (\alpha q - \beta q')(p'-q)^n}. \tag{10}$$

5. Generalizations. In the preceding examples we had a message capable of two forms transmitted in successive steps. It is easy to generalize this scheme to the case where the message can assume N different forms $1, 2, \cdots, N$. (For example, the message may be a digit which is repeatedly copied and is subject to copying errors.) The preceding theory applies, except that the matrix $P=(p_{jk})$ of transition probabilities is now of order N, and that more questions can be asked. We are, in this way, led to the general Markov chain with N possible states and constant (or stationary) transition probabilities. Finally, if we admit that the transition probabilities vary from step to step (variable proneness to lie) then we are led to the most general Markov chain with finitely many states.*

* For the general theory cf. Chapters 15 and 16 of An Introduction to Mathematical Probability and Its Applications by W. Feller (Wiley, New York, 1950).

A SIMPLE ITERATIVE SOLUTION FOR CERTAIN SIMULTANEOUS QUADRATIC EQUATIONS

W. S. LOUD, University of Minnesota

This short paper describes an iterative solution for a class of simultaneous quadratic equations. It shows how the non-linearity of the equations places a severe restriction on the possibility of the process. It also illustrated a case for which, once the possibility of the iterative solution is known, a starting point can easily be found from which the process always converges.

An iterative solution of the system,

$$x = f(x, y), \qquad y = g(x, y),$$

involves the selection of an initial point (x_1, y_1), and the determination of a succession of points (x_n, y_n) by the formulas

$$x_{n+1} = f(x_n, y_n), \qquad y_{n+1} = g(x_{n+1}, y_n) = h(x_n, y_n).$$

It is known that if (x_0, y_0) is a solution of the system, the process will converge to (x_0, y_0) if (x_1, y_1) is chosen sufficiently close to (x_0, y_0) and if both of the characteristic values of the matrix of the Jacobian

$$\frac{\partial(f, h)}{\partial(x, y)}$$

are less than unity in magnitude in a neighborhood of (x_0, y_0).

We shall consider the case where f is a quadratic function of y alone and g is a quadratic function of x alone. Without loss of generality we may consider the system,

$$y = x^2 + A, \qquad x = y^2 + B, \tag{1}$$

where A and B are constants.

For the system (1) the convergence condition is $|4x_0y_0| < 1$ where (x_0, y_0) is a real solution of the system.

To determine the conditions on A and B which will imply this condition, it is helpful to consider an A, B-plane and to analyze the system (1) in terms of regions in this plane. There are five possibilities for roots of (1): four real and distinct, four real with two coincident, two real and distinct and two complex, two real and coincident and two complex, and four complex. If there are two coincident roots, the pair always lies on the hyperbola $4xy = 1$. To find the points in the A, B-plane which give rise to this, we eliminate x and y from the three equations $4xy = 1$ and the system (1). The result is:

$$F_1(A, B) = 256A^2B^2 + 256A^3 + 256B^3 + 288AB - 27 = 0.$$

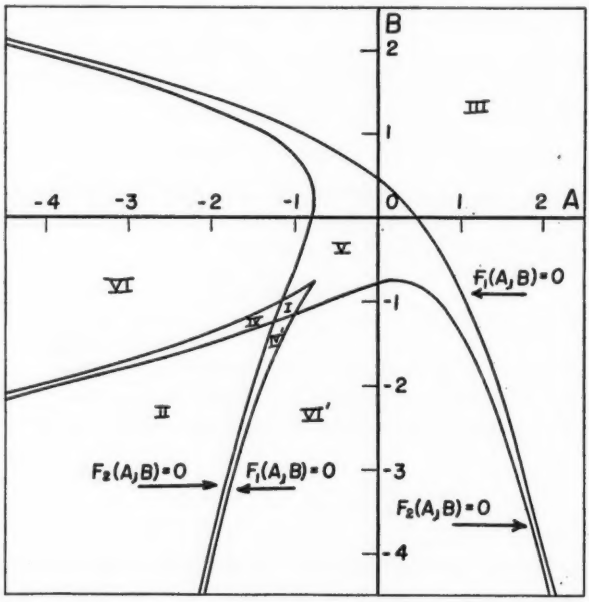
If $F_1(A, B)$ is not zero, we will have one of the three cases with no coincident roots.

The locus of $F_1 = 0$ in the A, B-plane will also be a boundary of the region

in that plane of values of A and B for which the convergence condition will be satisfied. The other boundary will be the locus of those values of A and B for which (1) has a root on the hyperbola $4xy=-1$. By the same process as above we find:

$$F_2(A, B) = 256A^2B^2 + 256A^3 + 256B^3 + 160AB + 125 = 0.$$

In the accompanying figure, the loci of $F_1=0$ and $F_2=0$ are plotted in the A, B-plane. There are formed eight regions, and the system (1) has a root for which the process converges if (A, B) lies in one of a certain four of these regions.



Region	Sign of F_1	Sign of F_2	Number of Real Roots	Number of Roots for which Iterative Process Converges
I	+	+	4	2
II	+	+	4	0
III	+	+	0	0
IV–IV′	+	−	4	1
V	−	+	2	1
VI–VI′	−	−	2	0

It will be necessary to know the regions of the x, y-plane in which the roots of (1) will fall for the various cases. It is interesting that the function F_1 intro-

duced earlier also plays a role here. The two branches of the curve $F_1(-x/2, -y/2)=0$ and the two branches of the hyperbola $4xy=1$ divide the x, y-plane into six regions. (The lower left branches are tangent at $(-1/2, -1/2)$.) If the system (1) has two distinct real roots and two complex, there will be one real root in each of the two regions between the branches of $F_1(-x/2, -y/2)=0$. If there are four distinct real roots, one will lie in each of the four remaining regions.

We now consider the problem of determining a starting point for the iterative process. The principal result is: If (x_0, y_0) is a root of the system (1) and $|4x_0y_0| < 1$, then at least one of the points $(0, A)$ and $(B, 0)$, the vertices of the graphs of the two equations, may be used as the starting point of an iterative process as described, and this process will converge to (x_0, y_0).

It should be remarked that if the system has two "convergent" roots, the iterative processes starting at the two vertices both converge, one to each of the roots.

To prove the above, two cases must be considered, that where the product x_0y_0 is non-negative, and that where it is negative. If the root lies on one of the axes, the process terminates immediately. If the root lies in the first or third quadrant of the x, y-plane, we may suppose without loss of generality that $|x_0| \leqq |y_0|$. If the process is started at $(0, A)$ it is not difficult to show that the values $0, x_1, x_2, \cdots$ form a monotonic, bounded sequence whose only limit can be x_0. This fact is immediately suggested if a diagram of the process is drawn. The values $y_1, y_2, \cdots$ must also converge to y_0. If $|x_0| \geqq |y_0|$, identical reasoning, with starting-point $(B, 0)$, establishes convergence in that case.

If the point (x_0, y_0) lies in the second or fourth quadrant, we do not have monotonic sequences. This fact can be seen from a diagram in which the points $(x_1, y_1), (x_2, y_1), (x_2, y_2), \cdots$ are joined by straight lines. A rectilinear "spiral" is formed, which will converge to (x_0, y_0) if (x_1, y_1) is sufficiently close to (x_0, y_0). We must show that one of the vertices $(0, A)$ and $(B, 0)$ is indeed sufficiently close. Let P_1 be a point in the second quadrant from which the process converges to a root in that quadrant. If the spiral through P_1 is followed outward, one of two things can occur. First the spiral may cross one of the axes. If this is the case, it is clear from a diagram that the spiral starting from the vertex on that axis will wind in between the turns of the spiral through P_1 and so will converge to the root (x_0, y_0). The other possibility is that the spiral through P_1, if followed outward, will converge to a limiting rectangle lying entirely within the second quadrant. If this were the case, a spiral starting from a vertex could never enter this rectangle, and could not converge to (x_0, y_0). It is clear that such a rectangle must have all four vertices on the two parabolas (1). It is not difficult to establish that the center of such a rectangle must lie on the hyperbola $4xy=-1$. It then follows, after some calculation, that the root (x_0, y_0) within the rectangle (and the only root in the second quadrant) must have $4x_0y_0 < -1$, so it is not a root to which the iterative process would converge. Thus, if (x_0, y_0) is a convergent root, the iterative process, when initiated from a

vertex, will converge to it.

The convergence criterion cited earlier, though well known, does not seem to be readily available in present-day literature. It is used, for example, by Levinson in the theory of non-linear differential equations.* The theory of iterative processes is old, dating back to 1674,** so it is very likely that the criterion involving the characteristic roots of the Jacobian appears in older literature. The convergence condition given in Scarborough's *Numerical Mathematical Analysis* for the iterative process

$$x_{n+1} = f(x_n, y_n), \qquad y_{n+1} = h(x_n, y_n), \tag{2}$$

is that, in the neighborhood of a solution (x_0, y_0),

$$|f_x| + |h_x| < 1 \quad \text{and} \quad |f_y| + |h_y| < 1.$$

The quantities involved need not be real. This condition is more stringent than necessary. This same condition is given in Cassini's *Calcoli Numerici Grafici E Meccanici.*

Although the Jacobian condition does not imply the above, it is always possible to reduce the general case, in which the Jacobian condition is satisfied, to a special case, in which the above condition is satisfied by means of a linear transformation of variables.

Without loss of generality, we may take the solution to be found by iteration as (0, 0). Let the characteristic values of the Jacobian matrix of f and h both be less than 1 in magnitude at (0, 0) and have continuity in a neighborhood. Then by a suitable (not necessarily real) transformation,

$$x = au + bv, \qquad y = cu + dv,$$

the system (2) will become

$$u_{n+1} = F(u_n, v_n), \qquad v_{n+1} = H(u_n, v_n), \tag{3}$$

where the Jacobian matrix of F and H will have at (0, 0) the same characteristic values, but will be in a *canonical form*, either

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \lambda & 0 \\ \epsilon & \lambda \end{pmatrix}.$$

The λ's are the characteristic values in question. The second form may occur if they are equal, but the quantity ϵ may be chosen arbitrarily small, so that $|\lambda| + |\epsilon| < 1$. With the hypothesis that the λ's are less than 1 in magnitude it is clear that (3) satisfies the convergence condition of Scarborough. Thus the u's and v's converge, so the x's and y's do also.

* Levinson: Transformation Theory of Non-Linear Differential Equations of the Second Order, Annals of Math. Vol. 45 (1944). See page 729.

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MATHEMATICAL NOTES

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A NOTE ON CROSS RATIO*

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Let F be an arbitrary field and m the projective line over F. Assuming further that F is not of characteristic 2, one sees readily that if τ is any 1-1 mapping of m onto itself that fixes 0, 1 and ∞ and preserves cross ratio -1 (*i.e.*, preserves harmonic sets), then τ is an automorphism of F. It is natural to inquire (*i*) whether it is possible to substitute some other number for -1 in the hypothesis so that the conclusion remains valid, and (*ii*) whether some more general statement can be made which will include the case of characteristic 2.

The purpose of this note is to answer these questions by the following theorem, valid for F of arbitrary characteristic. We denote τx by x'.

THEOREM. *If τ is any $1-1$ mapping of m onto itself leaving 0, 1 and ∞ fixed such that for some $k \in F$, $k \neq 0, 1$, τ preserves cross ratio k, then τ is an automorphism of F.*

We first prove a simple lemma that will be useful later.

LEMMA. *If $a \in F$ is left fixed by every τ satisfying the hypothesis of the theorem, then τ preserves cross ratio a.*

Proof. Let $R(x_1, x_2, x_3, x_4) = a$ and let ρ be the projective transformation that takes x_1 into ∞, x_2 into 0, and x_3 into 1. Then $\rho x_4 = a$. Let σ be the projective transformation such that $\sigma: x_1' \to \infty$, $x_2' \to 0$, $x_3' \to 1$. We are required to show that $\sigma x_4' = a$. Now $\sigma\tau\rho^{-1}$ fixes ∞, 0 and 1, it is 1–1, and it preserves cross ratio k, so by hypothesis $\sigma\tau\rho^{-1}a = a$. But $\sigma\tau\rho^{-1}a = \sigma x_4'$.

Now to prove the theorem. Since $R(\infty, 0, -x/k, -x) = k$, it follows that

$$(-x/k)' = (-x)'/k. \tag{1}$$

Since $R(x_1, x_2, x_3, x_4) = k$ if and only if $R(x_1, x_3, x_2, x_4) = 1 - k$, it follows that τ

* Written while the author was working under a grant from ONR.

preserves cross ratio $1-k$. It follows, in view of (1) and the equation $R(0, -x/k, x/(1-k), \infty)=k$, that

$$(-x)' = -x'. \tag{2}$$

To show that $(x+y)'=x'+y'$: If x or y is 0, this is immediate. If $y=-x/k$, it is a consequence of (1), (2), and $R(\infty, -x/k, 0, x-(x/k))=k$. If none of these relations between x and y hold, it follows from $R(y, -x/k, (x+y)/(1-k), \infty)=k$.

By (1), induction, repeated use of the lemma and the fact that τ is an automorphism of the additive group of F, it follows that τ preserves any cross ratio in K, where K is the subfield of F generated by k. It is worth remarking that since $-1\in K$ the theorem now follows, in case the characteristic of F is not 2, from the fact mentioned in the introduction. But a proof for the general case is not much longer.

To prove that τ is an automorphism of the multiplicative group of F, we first show that $(1/x)'=1/x'$. This is obtained from $R(x-k+1, 0, 1, k+(k-k^2)/x)=k$. $(x^2)'=(x')^2$ follows from $R(x^2-2kx+k^2, 1, x-k, 1+kx-2k^2+(k^3-k)/x)=k$. In each of these cases, two elements inside the cross ratio symbol can coincide only if $x\in K$, where the assertions to be proven are already known to hold.

Next, $((xy)')^2=(x'y')^2$ follows from $R((1/x^2)+y-ky, 0, y, ky+k(1-k)\cdot(xy)^2)=k$, provided all entries are distinct. But each of the cases in which the entries are not distinct can easily be handled separately. Hence, $(xy)'=\pm x'y'$. If F is of characteristic 2, we are finished. Otherwise, if for some $xy\neq 0$, we have $(xy)'=-x'y'$, then $\pm(x'y'+x')=(x(y+1))'=(xy+x)'=-x'y'+x'$, which is impossible.

COROLLARY. *If τ is any 1-1 mapping onto itself such that for some $k\in F$, $k\neq 0, 1$, τ preserves cross ratio k, then τ is the composition of a projective transformation of m and an automorphism of F.*

THE ROW-SUMS OF THE INVERSE MATRIX

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This note closes with an unsolved problem, and begins with the following surprising, yet almost trivial, fact about a matrix with an inverse.

If the sum of the elements in each row of a square matrix is k, then the sum of the elements in each row of the inverse matrix is $1/k$.

The proof is as follows. Let A be $m\times m$, non-singular, with the stated property. Let B be its inverse. Then for $n\leqq m$,

$$1 = \sum_{r=1}^{m} \delta_{nr} = \sum_{r=1}^{m}\sum_{s=1}^{m} b_{ns}a_{sr} = \sum_{s=1}^{m}\sum_{r=1}^{m} b_{ns}a_{sr} = k\sum_{s=1}^{m} b_{ns}.$$

This completes the proof. (A is singular if $k=0$.)

The result has an obvious application to magic squares.

Since the crucial point is an interchange of summations we can conclude:

The same is true for an infinite matrix A such that the inverse matrix has finite rows (*i.e.* almost all the elements in each row are zero). *Furthermore, the inverse matrix need be only a left inverse.*

A triangular matrix, *i.e.* one such that the nth row has no non-zero entries beyond the nth place, can be shown to have a two-sided inverse if and only if it is normal, *i.e.* has no zeros on its main diagonal. Such a matrix is easily shown to have a normal inverse. Thus we conclude:

The result holds for normal matrices.

We observe that the row-sums of the inverse need not exist. For example, let A have the rows $(1, 1, 0, 0, 0, \cdots)$, $(0, 1, 1, 0, 0, 0, \cdots)$, $(0, 0, 1, 1, 0, 0, 0, \cdots)$, $\cdots$, then the two-sided inverse has the rows $(1, -1, 1, -1, \cdots)$, $(0, 1, -1, 1, -1, \cdots) \cdots$. We notice, however, the suggestive fact that the rows of the inverse matrix are summable by many of the classical methods to the expected value $\frac{1}{2}$. That this is not generally true is shown by replacing, in the nth row of A as defined above, the pair $(1, 1)$ by (a, b) or by (b, a) respectively according as n is odd or even, where a, b are any numbers satisfying $ab \neq 0$.

A further observation is that it is not sufficient to assume only that the inverse is a right inverse. For let A be formed by adjoining a column of zeros on the left side of the identity matrix. It has, as right inverse, the matrix obtained by adjoining a row of zeros to the top of the identity matrix. The row-sums of A are one, not so for the right inverse.

I conjecture that "no" is the answer to the Problem: *Will the result hold for a matrix A and its inverse* (or left inverse) *if it is merely assumed that each row of the inverse matrix has a sum?*

REPRESENTATION OF THE INTEGERS BY POSITIVE INTEGERS

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The integers are commonly constructed from the positive integers by the use of pairs of positive integers. That is, one considers the integers as pairs* (a, b) for which

$$(1) \qquad \begin{aligned} (a, b) &= (c, d) \quad \text{if} \quad a + d = b + c, \\ (a, b) + (c, d) &= (a + c, b + d), \\ (a, b) \times (c, d) &= (ac + bd, ad + bc), \end{aligned}$$

for positive integers a, b, c, and d. From (1) and the properties of positive integers it is then possible to establish the commutative, associative, distributive laws, *etc.* for integers.

We propose here to consider the integers as single positive integers. To do this we consider the set of positive integers as divided up into two classes, the even numbers (the negative integers) and the odd numbers (the non-negative integers). Addition ($\oplus$) is then defined by

* More exactly, as classes of equivalent pairs.

$$
\begin{aligned}
&m \oplus n = m + n \quad \text{if} \quad m \text{ and } n \text{ are both even,} \\
&m \oplus n = m + n - 1 \quad \text{if} \quad m \text{ and } n \text{ are both odd,} \\
&m \oplus n = n \oplus m = m - n + 1 \quad \text{if} \quad m \text{ is even, } n \text{ odd, and } m > n, \\
&m \oplus n = n \oplus m = m - n \quad \text{if} \quad m \text{ is odd, } n \text{ even, and } m > n.
\end{aligned} \tag{2}
$$

Here, of course, the addition on the right is the ordinary addition of positive integers while $m-n$ is defined by $m-n=c$ (m, n, c positive integers) if $m=n+c$. Likewise $m>n$ means there exists a positive integer k such that $m=n+k$.

Similarly, multiplication ($\otimes$) is defined by

$$
\begin{aligned}
&m \otimes n = (mn + 2)/2 \quad \text{if} \quad m \text{ and } n \text{ are both even,} \\
&m \otimes n = (mn - m - n + 3)/2 \quad \text{if} \quad m \text{ and } n \text{ are both odd,} \\
&m \otimes n = n \otimes m = (nm - m)/2 \quad \text{if} \quad m \text{ is even, } n \text{ odd, and } n > 1, \\
&1 \otimes n = n \otimes 1 = 1 \quad \text{for all positive integers } n.
\end{aligned} \tag{3}
$$

Here the multiplication on the right is the ordinary multiplication of positive integers.

We now claim that the set of positive integers under $\oplus$ and $\otimes$ is isomorphic to the set of pairs of positive integers (*i.e.*, the integers) under the correspondence

$$(a, b) \leftrightarrow (2a + 1) \oplus 2b. \tag{4}$$

By virtue of $(2)_4$ and $(2)_3$ respectively, (4) can be re-written as

$$
\begin{aligned}
&(a, b) \leftrightarrow (2a + 1) - 2b = 2(a - b) + 1^* \quad \text{if} \quad a \geqq b, \\
&(a, b) \leftrightarrow 2b - (2a + 1) + 1 = 2(b - a) \quad \text{if} \quad a < b.
\end{aligned} \tag{5}
$$

For example, $(4, 1)\leftrightarrow 2(4-1)+1=7$ and $(2, 6)\leftrightarrow 2(6-2)=8$. Then $(4, 1)+(2, 6)=(4+2, 1+6)=(6, 7)$ by $(1)_2$. But $7\oplus 8=8-7+1=2$ by $(2)_3$ and $(6, 7)\leftrightarrow 2(7-6)=2$. Similarly, $(4, 1)\times(2, 6)=(4\cdot 2+1\cdot 6, 4\cdot 6+1\cdot 2)=(14, 26)$ by $(1)_3$. But $7\otimes 8=(7\cdot 8-8)/2=24$ by $(3)_3$ and $(14, 26)\leftrightarrow 2(26-14)=24$.

The proof of this isomorphism is straightforward but somewhat tedious due to the complexity of the definitions of $\oplus$ and $\otimes$ and will be left to the interested reader.

A PRIME-REPRESENTING FUNCTION

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Recently Mills [1] proved that there is a fixed number A such that, for all positive integral values of x, the number

$$[A^{3^x}]$$

is a prime. Here $[y]$ denotes as usual the greatest integer $\leqq y$. The proof de-

* If $a=b$, $a-b$ is not a positive integer and we should consider $(2b+1)-2b=1$. We write $2(a-b)+1$ for brevity and understand that $2(b-b)+1=1$ without defining $b-b$.

pends on the result that

$$p_{n+1} - p_n < K p_n^{5/8},$$

where K is a fixed positive number and p_n is the nth prime. This result (due to Ingham [2]) is fairly deep.

Here I prove

THEOREM *There is a number α such that, if*

$$g_0 = \alpha, \qquad g_{n+1} = 2^{g_n}, \qquad (n \geqq 0),$$

then

$$[g_n] = [2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}], \qquad (n \geqq 1),$$

is always a prime.

I require only the very elementary result that, for every $N \geqq 2$, there is a prime between N and $2N$ (see [2]). By this, we can choose a sequence of primes $\{P_n\}$ such that $P_1 = 2$ or 3 and

$$2^{P_n} < P_{n+1} < P_{n+1} + 1 < 2^{P_{n+1}}.$$

We take all logarithms to base 2, and write,

$$u_n = \log^{(n)} P_n, \qquad v_n = \log^{(n)} (P_n + 1),$$

where $\log^{(n)}$ denotes the nth iterate of the logarithm. We have

$$P_n < \log P_{n+1} < \log (P_{n+1} + 1) < P_n + 1$$

and so

$$u_n < u_{n+1} < v_{n+1} < v_n.$$

Hence $u_n \to \alpha$ (say) as $n \to \infty$ and

$$u_n < \alpha < v_n, \qquad P_n < g_n < P_n + 1$$

and so

$$P_n = [g_n].$$

There are, of course, any number of possible values of α. For example, if

$$\alpha = 1.9287800 \cdots,$$

we have

$$P_1 = 3, \qquad P_2 = 13, \qquad P_3 = 16381$$

and P_4 has some 5000 digits.

If we use only the almost trivial result that there is a number B such that, for every $N \geqq 2$, there is a prime between N and BN (see [3]), we can prove that,

for some β, $[h_n]$ is always a prime, where

$$h_0 = \beta, \qquad h_{n+1} = B^{h_n} \qquad (n \geqq 0).$$

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ON THE DENSITY OF SETS OF GAUSSIAN INTEGERS

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1. Introduction. We present two theorems and a remark on the density of sets of Gaussian integers in the first quadrant of the complex plane. A Gaussian integer is a complex number of the form $a+a'i$ where a and a' are integers. Attention is confined exclusively to the set Q of those Gaussian integers $a+a'i$ with $a \geqq 0$ and $a' \geqq 0$ but not both $a=0$ and $a'=0$.

If S is a set of positive integers, Schnirelman [1] lets $S(x)$ denote the number of elements s of S with $s \leqq x$ and defines the density of S to be the greatest lower bound of the fraction $S(x)/x$ for $x=1, 2, \cdots$. By analogy, we adopt the following.

Definition: Let A be a set of Gaussian integers (subset of Q). For each Gaussian integer $x+yi$, let $A(x+yi)$ denote the number of Gaussian integers $a+a'i$ in A with $a \leqq x$ and $a' \leqq y$. The density of A is defined to be

$$\text{g.l.b.} \frac{A(x+yi)}{xy+x+y}$$

taken over all $x+yi$ in Q.

2. Two theorems. If A and B are subsets of Q, the sum $A+B$ is defined to be the set C consisting of all elements $a+a'i$, $b+b'i$ and all $a+b+(a'+b')i$ with $a+a'i$ in A and $b+b'i$ in B. Let the densities of A, B, and C be α, β and γ respectively.

THEOREM 1. *If* $\alpha+\beta \geqq 1$, *then* $\gamma=1$.

Proof. Suppose $\gamma<1$, then there is an element $u+vi$ not in C. Whenever $a+a'i$ is in A, $u-a+(v-a')i$ is not in B. Therefore $A(u+vi)+B(u+vi) \leqq uv+u+v-1$. Thus

$$\frac{A(u+vi)+B(u+vi)}{uv+u+v} < 1.$$

That is, $\alpha+\beta<1$. This contradicts the hypothesis.

THEOREM 2. *If B contains all numbers ji for $j=1, 2, \cdots$, then* $\gamma \geqq \alpha_0+\beta-\alpha_0\beta$, where

$$\alpha_0 = \text{g.l.b.}_x \frac{A(x+0i)}{x}.$$

Proof. Take any number $x+yi$ in Q. Define B_j to be the set of all elements $b+ji$ whenever it is in B, and $B_j(x)$ denotes the number of elements $b+ji$ in B_j with $b \leqq x$. Similarly define A_j and C_j. Let $k_{j1}, k_{j2}, \cdots, k_{jn_j}$ be all the numbers less than x such that $k_{jt}+ji$ is in B_j but $k_{jt}+1+ji$ is not in B_j, $t=1, 2, \cdots, n_j$. Let $m_{j1}, m_{j2}, \cdots, m_{jn_j}$ be the lengths of corresponding gaps such that no element $u+ji$ in the range $k_{jt}<u<k_{jt}+m_{jt}$ belongs to B_j, but $k_{jt}+m_{jt}+1+ji$ belongs to B_j, and define $m_{jn_j}=x-k_{jn_j}$ if $x+ji$ is not in B_j. When

$$j = 0, \qquad B_0(x) = x - \sum_{t=1}^{n_0} m_{0t},$$

and

$$\begin{aligned} C_0(x) &\geqq B_0(x) + \sum_{t=1}^{n_0} A_0(m_{0t}) \geqq B_0(x) + \alpha_0 \sum_{t=1}^{n_0} m_{0t} \\ &= B_0(x) + \alpha_0(x - B_0(x)) = (1-\alpha_0)B_0(x) + \alpha_0 x. \end{aligned}$$

When $j>0$, we have

$$B_j(x) = x + 1 - \sum_{t=1}^{n_j} m_{jt},$$

and then

$$\begin{aligned} C_j(x) &\geqq B_j(x) + \sum_{t=1}^{n_j} A_0(m_{jt}) \geqq B_j(x) + \alpha_0 \sum_{t=1}^{n_j} m_{jt} \\ &= B_j(x) + \alpha_0(x + 1 - B_j(x)) \\ &= (1-\alpha_0)B_j(x) + \alpha_0(x+1). \end{aligned}$$

Hence

$$\begin{aligned} C(x+yi) = \sum_{j=0}^{y} C_j(x) &\geqq (1-\alpha_0)\sum_{j=0}^{y} B_j(x) + \alpha_0 x + \alpha_0 y(x+1) \\ &= (1-\alpha_0)B(x+yi) + \alpha_0(xy+x+y) \\ &\geqq (1-\alpha_0)\beta(xy+x+y) + \alpha_0(xy+x+y). \end{aligned}$$

Therefore

$$\frac{C(x+yi)}{xy+x+y} \geqq (1-\alpha_0)\beta + \alpha_0 = \alpha_0 + \beta - \alpha_0\beta.$$

3. Remark. Let A_0, B_0 be sets of positive integers with densities α_0 and β_0 respectively, and let $A_0+B_0=C_0$ have density γ_0. Mann [2] proved that $\gamma_0 \geqq \alpha_0+\beta_0$ or 1. By our definition and the two theorems proved in section 2 one might conjecture that Mann's theorem also holds in the present case of two dimensions. But, however, this is not true as shown by the following example: let A be the set consisting of numbers 1, 2, 5, 6, 9, 10, 13, 14, 17, 18, i, $1+i$, $4+i$, $5+i$, $8+i$, $9+i$, $12+i$, $13+i$, $16+i$, $17+i$, $20+i$, and all numbers $a+a'i$ with $a>20$, or $a'>1$; and let B be the set consisting of the numbers 1, 4, 5, 8, 9, 13, 17, i, $3+i$, $4+i$, $8+i$, $12+i$, $16+i$, $20+i$, and all $b+b'i$ with $b>20$ or $b'>1$. C then is the set consisting of all numbers in A, all numbers in B and numbers 3, 7, 11, 15, 19, $2+i$, $6+i$, $10+i$, $14+i$, and $18+i$. It is readily calculated that $\alpha=1/2$, $\beta=1/3$ and $\gamma=34/41$ which is less than $\alpha+\beta$.

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ON FEUERBACH'S THEOREM*

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1. A general theorem. *If the distances O_2O_3, O_3O_1, O_1O_2 between the centers of three circles (O_1), (O_2), (O_3), having radii R_1, R_2, R_3, satisfy a relation of the form*

$$(1) \qquad (O_2O_3)R_1 \pm (O_3O_1)R_2 \pm (O_1O_2)R_3 = 0,$$

then the circumcircle (O) of triangle $O_1O_2O_3$ is tangent to the radical circle (O') of the circles (O_1), (O_2), (O_3), and the radical center O' is at the center of one of the tritangent circles of the triangle determined by the radical axes of circle (O) in association with each of the circles (O_1), (O_2), (O_3).

The locus of points the ratio of whose distances from O_2 and O_3 is equal to R_2/R_3 is a circle Γ_{23} having for diameter the segment joining the centers of homothety of circles (O_2) and (O_3), and Γ_{23} belongs to the pencil of circles defined by (O_2) and (O_3). Circle Γ_{23} and the two analogous circles Γ_{31} and Γ_{12} have, in general, two common points P and P'. Since circle Γ_{23}, for example, is orthogonal to all circles passing through O_2 and O_3, it follows that the three circles Γ_{23}, Γ_{31}, Γ_{12} are orthogonal to circle (O), of radius R. Also, since circle Γ_{23} belongs to the pencil of circles determined by circles (O_2) and (O_3), it cuts orthogonally all circles orthogonal to both (O_2) and (O_3). Thus the three circles Γ_{23}, Γ_{31}, Γ_{12} are also orthogonal to the radical circle (O') of the three circles (O_1), (O_2), (O_3). It follows that the points P and P' are inverses of each other in

* Translated from the French by Howard Eves.

both circle (O) and circle (O'). But the distances of P and P' from the vertices of triangle $O_1O_2O_3$ are proportional to R_1, R_2, R_3. Therefore, by (1),

$$(O_1P)(O_2O_3) \pm (O_2P)(O_3O_1) \pm (O_3P)(O_1O_2) = 0,$$

and P (and similarly P') lies on circle (O). Points P and P' are then coincident, and circle (O) is tangent to circle (O').

On the other hand, if we denote by d_1, d_2, d_3 the algebraic distances of O' from the radical axes of the pairs of circles (O) and (O_1), (O) and (O_2), (O) and (O_3), we find that $d_1 = d_2 = d_3$, for

$$\begin{aligned}(O'O)^2 - R^2 - (O'O_1)^2 + R_1^2 &= (O'O)^2 - R^2 - (O'O_2)^2 + R_2^2 \\ &= (O'O)^2 - R^2 - (O'O_3)^2 + R_3^2 \\ &= 2(O'O)d_1 = 2(O'O)d_2 = 2(O'O)d_3.\end{aligned}$$

This completes the proof of the theorem.

2. Applications. Let (O) be the circumcircle of a triangle $T \equiv ABC$; A_1, B_1, C_1 and A', B', C' the midpoints of the sides BC, CA, AB, and the feet of the corresponding altitudes; (N) the nine point circle; and D, E, F the points of contact of one of the tritangent circles, (I), with sides BC, CA, AB.

(A) *Feuerbach's Theorem. The radical circle of the three circles* (A_1, A_1D), (B_1, B_1E), (C_1, C_1F) *clearly coincides with circle* (I), *and since*

$$(B_1C_1)(A_1D) \pm (C_1A_1)(B_1E) \pm (A_1B_1)(C_1F) = 0,$$

circles (I) *and* (N) *are tangent to each other.*

(B) *In a triangle* T, *the radical circle* (O') *of the three circles having for centers the midpoints of the sides of* T *and passing through the feet of the corresponding altitudes is one of the tritangent circles of the anticomplementary triangle* $T'' = A''B''C''$ *of the orthic triangle* $T' = A'B'C'$ *of* T.

The evident relation

$$(B_1C_1)(A_1A') \pm (C_1A_1)(B_1B') \pm (A_1B_1)(C_1C') = 0 \tag{2}$$

shows immediately that the radical circle (O') of the three circles $(O_1) \equiv (A_1, A_1A')$, $(O_2) \equiv (B_1, B_1B')$, $(O_3) \equiv (C_1, C_1C')$ is concentric with one of the tritangent circles of the triangle determined by the radical axes of the circles (N) and (O_1), (N) and (O_2), (N) and (O_3). Now these axes pass through A', B', C' and are perpendicular to NA_1, NB_1, NC_1, and, consequently, to the lines OA, OB, OC, which proves that the triangle which they determine is the triangle T''. Since triangles T and T'' have the same nine point circle (N), it follows, from (1) and (2), that (N) is tangent to (O').

If T is acute angled, (O') is interior to (N), and if T is obtuse angled, (O') is exterior to (N). In either case, by applying Feuerbach's theorem to triangle T'', we see that (O') is one of the tritangent circles of T''. If T is right angled at

A, then circle (O') reduces to the point A'.

3. Additional remarks. If T is acute angled, the radius of circle (O') is given by

$$\rho = 4R \cos A \cos B \cos C,$$

and if T is obtuse angled, at A say,

$$\rho = (2S \cot A)/R,$$

where S denotes the area of the fundamental triangle. The point of contact of circles (N) and (O') is one of the Feuerbach points of triangle T''. We have given other properties of this point. See this MONTHLY, [1947, p. 448].

CLASSROOM NOTES

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COMPLEX NUMBERS AND VECTOR ALGEBRA

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1. Introduction. Vectors in the plane are frequently represented by complex numbers and treated by the corresponding rules. However, in many applications they are treated by vector algebra and multiplied to form vector and scalar products. The student is sometimes puzzled about the relation between these two schemes of representation. The literature appears to contain no explicit discussion of this point. The present paper supplies this lack and also presents complex numbers in a somewhat new light.

Let us imagine that we are faced with the task of inventing an algebra of directed quantities, that is, quantities representable by arrows or vectors in the plane of the paper. By parallel displacement, any such vector may be drawn from the origin of a rectangular coordinate system to some point P in the plane (see Figure 1). It is natural to use letters, say α, β, γ, $\cdots$, to represent such vectors and to try to give meanings to sums, differences and products like $\alpha+\beta$, $\alpha-\beta$, $\alpha\beta$. Keeping in mind simple physical applications like the composition of forces or velocities, it is easy to arrive at the usual geometric interpretations of the addition and subtraction of two vectors and of the product of a vector by a scalar. The corresponding algebra is also simple.

Let $\mathbf{H}$ denote a unit vector along the horizontal axis and $\mathbf{V}$ a unit vector along the vertical axis (see Figure 1). In terms of vector addition and of mul-

tiplication by a scalar, we may represent an arbitrary vector in the form

$$\alpha = a_1\mathbf{H} + a_2\mathbf{V}$$

with real numbers a_1 and a_2. We may also define the magnitude of α, written $|\alpha|$, as $\sqrt{a_1^2+a_2^2}$.

The difficulty is to find an appropriate definition for the product of two vectors α and β. It is natural to attempt to define the product in such a way that magnitudes shall multiply, that is, so that $|\alpha\beta| = |\alpha|\,|\beta|$. This stipulation leads to an essentially unique result if we add the requirement that $\mathbf{H}$ shall act

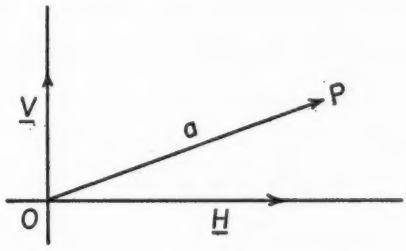

FIG. 1

like unity, so that specifically.

$$\mathbf{HV} = \mathbf{VH}(=\mathbf{V}) \quad \text{and} \quad \mathbf{HH} = \mathbf{H}\ (\text{i.e., } \mathbf{H}^2 = \mathbf{H}).$$

We proceed to derive this unique result.

To begin with, it is necessary to evaluate $\mathbf{V}^2$. Let

$$\mathbf{V}^2 = a\mathbf{H} + b\mathbf{V}$$

where a and b are to be determined.

Since $|\mathbf{V}| = 1$, we have, equating the squares of the magnitudes,

$$a^2 + b^2 = 1.$$

Now consider

$$\begin{aligned}(\mathbf{H} + \mathbf{V})(\mathbf{H} - \mathbf{V}) &= \mathbf{H}^2 + \mathbf{VH} - \mathbf{HV} - \mathbf{V}^2 = \mathbf{H}^2 - \mathbf{V}^2 \\ &= \mathbf{H} - a\mathbf{H} - b\mathbf{V} = (1 - a)\mathbf{H} - b\mathbf{V}.\end{aligned}$$

Equating squares of magnitudes,

$$4 = (1 - a)^2 + b^2.$$

Combining with

$$1 = a^2 + b^2$$

we have

$$a = -1, \qquad b = 0.$$

Hence

$$\mathbf{V}^2 = -\mathbf{H}.$$

If

$$\alpha = a_1\mathbf{H} + a_2\mathbf{V}$$

and

$$\beta = b_1\mathbf{H} + b_2\mathbf{V}$$

are two vectors, their product is easily found to be

$$\alpha\beta = (a_1b_1 + a_2b_2)\mathbf{H} + (a_1b_2 + a_2b_1)\mathbf{V}. \tag{1}$$

Then

$$\begin{aligned} |\alpha\beta|^2 &= (a_1b_1 - a_2b_2)^2 + (a_1b_2 + a_2b_1)^2 \\ &= (a_1^2 + a_2^2)(b_1^2 + b_2^2) = |\alpha|^2|\beta|^2 \end{aligned}$$

for arbitrary a_1, a_2, b_1, b_2. Hence the stipulation $|\alpha\beta| = |\alpha||\beta|$ can be carried through in general.

The scheme just discussed is, except for notation, that of complex numbers. In fact, if we write $\alpha = a_1 + ib_1$ and $\beta = a_2 + ib_2$ and multiply as usual, replacing i^2 by -1, the result corresponds to (1). The scheme of complex numbers is therefore essentially the only one for which the magnitude of the product equals the product of the magnitudes.

2. Vector algebra. Vector algebra uses a somewhat different scheme of representation of vectors in the plane. The vector α of section 1 would be written

$$\alpha = a_1\mathbf{i} + a_2\mathbf{j}$$

replacing $\mathbf{H}$ by $\mathbf{i}$ and $\mathbf{V}$ by $\mathbf{j}$.

So long as one deals only with addition, subtraction and multiplication by scalars, no difference occurs other than this trivial notational one. But as soon as one speaks of multiplying vectors α and β, an essential difference comes in.

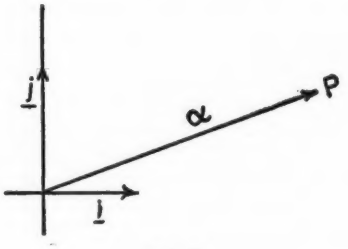

FIG. 2

As is well known, there are two types of product used in vector algebra, the scalar product and the vector product. If

$$\alpha = a_1\mathbf{i} + a_2\mathbf{j}$$
$$\beta = b_1\mathbf{i} + b_2\mathbf{j}$$

these products are

(2) $$\alpha \cdot \beta = a_1b_1 + a_2b_2 \qquad \text{(scalar product)}$$

(3) $$\alpha \times \beta = (a_1b_2 - a_2b_1)\mathbf{k} \qquad \text{(vector product)}$$

where $\mathbf{k}$ is a unit vector perpendicular to the plane of α and β.

The choice of these expressions to represent products is dictated by physical applications (work done by a force, moment of a force). Here we are concerned only with the formal relations among the definitions (1), (2) and (3).

The relations are the following.

$$\alpha \cdot \beta = \text{Re}\ (\bar{\alpha}\beta)$$
$$\alpha \times \beta = \text{Im}\ (\bar{\alpha}\beta)\mathbf{k}$$

where Re stands for "real part of" and Im for "imaginary part of" and where $\bar{\alpha}$ is the conjugate of α, that is, $a_1 - a_2 i$. In fact,

$$\bar{\alpha}\beta = (a_1b_1 + a_2b_2) + (a_1b_2 - b_1a_2)i,$$
$$\text{Re}\ (\bar{\alpha}\beta) = a_1b_1 + a_2b_2 = \alpha \cdot \beta,$$
$$\text{Im}\ (\bar{\alpha}\beta) = a_1b_2 - b_1a_2, \text{ the coefficient of } \mathbf{k} \text{ in } \alpha \times \beta.$$

To illustrate this relationship, consider two examples from geometry.

EXAMPLE 1. To prove that an angle inscribed in a semicircle is a right angle.

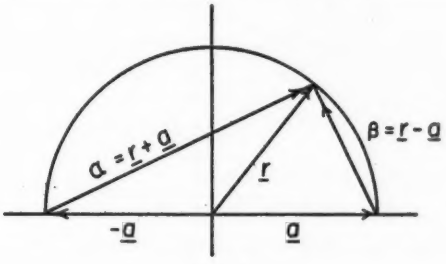

FIG. 3

Vector algebra proof:

$$\alpha \cdot \beta = (\mathbf{r} + \mathbf{a}) \cdot (\mathbf{r} - \mathbf{a})$$
$$= \mathbf{r}^2 - \mathbf{a}^2$$
$$= 0, \quad \text{since} \quad |\mathbf{r}| = |\mathbf{a}|.$$

Hence $\alpha \perp \beta$.

Complex number proof:

$$\begin{aligned}\bar{\alpha}\beta &= [(x+a)-iy][(x-a)+iy] \\ &= x^2 - a^2 + y^2 + 2aiy \\ &= 2aiy, \quad \text{since} \quad |x+iy| = a.\end{aligned}$$

$$\text{Re}\,(\bar{\alpha}\beta) = 0.$$

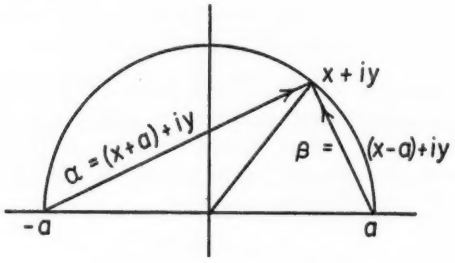

FIG. 4

Hence $\alpha \perp \beta$.

EXAMPLE 2. To derive the difference formulas of trigonometry.

Vector algebra proof:

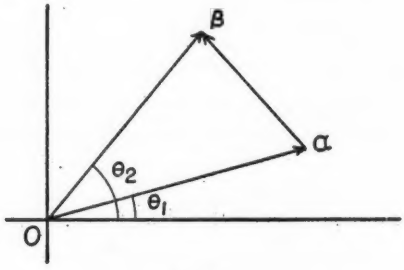

FIG. 5

$$\alpha = \cos\theta_1 \mathbf{i} + \sin\theta_1 \mathbf{j}$$

$$\beta = \cos\theta_2 \mathbf{i} + \sin\theta_2 \mathbf{j}$$

$$\alpha \cdot \beta = \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2.$$

But

$$\alpha \cdot \beta = |\alpha||\beta| \cos(\theta_2 - \theta_1) = \cos(\theta_2 - \theta_1).$$

Hence

$$\begin{aligned}\cos(\theta_2 - \theta_1) &= \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 \\ \alpha \times \beta &= (\sin\theta_2 \cos\theta_1 - \cos\theta_2 \sin\theta_1)\mathbf{k} \\ &= |\alpha||\beta| \sin(\theta_2 - \theta_1)\mathbf{k} = \sin(\theta_2 - \theta_1)\mathbf{k}.\end{aligned}$$

Hence

$$\sin(\theta_2 - \theta_1) = \sin\theta_2 \cos\theta_1 - \cos\theta_2 \sin\theta_1.$$

Note:

Since the formulas $\alpha\cdot\beta = |\alpha||\beta| \cos(\theta_2-\theta_1)$ and $\alpha\times\beta = |\alpha||\beta| \sin(\theta_2-\theta_1)\mathbf{k}$, used in this proof, are often established by way of the difference formulas, the proof may give the appearance of circularity. However, there is no difficulty in avoiding circularity. In fact,

$$(\beta - \alpha)^2 = \beta^2 + \alpha^2 - 2\alpha\cdot\beta.$$

By the law of cosines

$$(\beta - \alpha)^2 = |\beta - \alpha|^2 = \beta^2 + \alpha^2 - 2|\alpha||\beta| \cos(\theta_2 - \theta_1).$$

Equating these two results gives the first formula. The second follows by equating two expressions for the area of the triangle $O\alpha\beta$.

Complex number proof:

$$\alpha = \cos\theta_1 + \sin\theta_1 i$$

$$\beta = \cos\theta_2 + \sin\theta_2 i$$

$$\bar{\alpha}\beta = (\cos\theta_2\cos\theta_1 + \sin\theta_2\sin\theta_1) + (\sin\theta_2\cos\theta_1 - \cos\theta_2\sin\theta_1)i$$

$$= \cos(\theta_2 - \theta_1) + \sin(\theta_2 - \theta_1)i.$$

Conclusions as above.

Note that

$$\alpha\cdot\beta = \mathrm{Re}(\bar{\alpha}\beta), \qquad \alpha\times\beta = \mathrm{Im}(\bar{\alpha}\beta)\mathbf{k}.$$

3. Dimensions higher than two. It is of interest to show that there exists no method of multiplication which yields $|\alpha\beta| = |\alpha||\beta|$ for arbitrary vectors α and β in *three* dimensions, if we make the additional assumption that the associative law holds, $(\alpha\beta)\gamma = \alpha(\beta\gamma)$.

Let

$$\alpha = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3$$

where $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ are unit vectors along the axes of a rectangular coordinate system. Assume that $\mathbf{u}_1$ acts like unity so that

$$\mathbf{u}_1^2 = \mathbf{u}_1, \qquad \mathbf{u}_1\mathbf{u}_2 = \mathbf{u}_2\mathbf{u}_1(=\mathbf{u}_2), \qquad \mathbf{u}_1\mathbf{u}_3 = \mathbf{u}_3\mathbf{u}_1(=\mathbf{u}_3).$$

If

$$\mathbf{u}_2^2 = a\mathbf{u}_1 + b\mathbf{u}_2 + c\mathbf{u}_3$$

with undetermined a, b and c,

$$(\mathbf{u}_1 - \mathbf{u}_2)(\mathbf{u}_1 + \mathbf{u}_2) = \mathbf{u}_1^2 - \mathbf{u}_2^2 = \mathbf{u}_1 - (a\mathbf{u}_1 + b\mathbf{u}_2 + c\mathbf{u}_3)$$
$$= (1 - a)\mathbf{u}_1 - b\mathbf{u}_2 - c\mathbf{u}_3.$$

Equating squares of magnitudes

$$4 = (1 - a)^2 + b^2 + c^2.$$

Since $a^2+b^2+c^2=1$, we have

$$a = -1, \qquad b = c = 0.$$

Thus

$$\mathbf{u}_2^2 = -\mathbf{u}_1.$$

Similarly,

$$\mathbf{u}_3^2 = -\mathbf{u}_1.$$

Now let

$$\mathbf{u}_2\mathbf{u}_3 = r\mathbf{u}_1 + s\mathbf{u}_2 + t\mathbf{u}_3.$$

Then

$$\mathbf{u}_2(\mathbf{u}_2 + \mathbf{u}_3) = \mathbf{u}_2^2 + \mathbf{u}_2\mathbf{u}_3$$
$$= (r - 1)\mathbf{u}_1 + s\mathbf{u}_2 + t\mathbf{u}_3.$$

Equating squares of magnitudes

$$2 = (r - 1)^2 + s^2 + t^2.$$

Since $r^2+s^2+t^2=1$, $r=0$ and

$$\mathbf{u}_2\mathbf{u}_3 = s\mathbf{u}_2 + t\mathbf{u}_3.$$

Multiply on the left by $\mathbf{u}_2$. Since

$$\mathbf{u}_2(\mathbf{u}_2\mathbf{u}_3) = (\mathbf{u}_2^2)\mathbf{u}_3 = -\mathbf{u}_1\mathbf{u}_3 = -\mathbf{u}_3,$$
$$-\mathbf{u}_3 = -s\mathbf{u}_1 + t(s\mathbf{u}_2 + t\mathbf{u}_3).$$

Hence $s=0$, $t^2=-1$. This result is inconsistent with the fact that t must be real. It also conflicts with the necessary condition $t^2+s^2=1$.

For arbitrary vectors in 4 dimensions, it is possible to define the product so that magnitudes *do* multiply. In fact the methods of this section lead directly to quaternions which have this property. The same methods show that if multiplication is associative, the condition $|\alpha\beta| = |\alpha|\,|\beta|$ is impossible to satisfy for arbitrary vectors in a space of more than 4 dimensions. The proofs of both statements are well within the ability of any good student and should prove illuminating.

PARAMETRIC EQUATIONS AND PROPER INTERPRETATION OF MATHEMATICAL SYMBOLS

A. ERDÉLYI, California Institute of Technology

The point raised recently (this MONTHLY, vol. 58, pp. 106–107, 1951) by A. D. Fleshler is a pertinent one and deserves further discussion. The examples given by Fleshler show very clearly that mathematical symbols have no existence of their own: they are given a lease of life by an exact definition, and cannot function except in conjunction with that definition.

The chain of operations

$$\int_0^{4\pi} \sqrt{1-\cos x}\,dx = \sqrt{2}\int_0^{4\pi} \sin\frac{x}{2}\,dx = 0 \tag{1}$$

is reprehensible *not* because the result is wrong, but rather because the point of departure is not well defined. As a matter of fact, it is possible to advance a definition of $\sqrt{1-\cos x}$ which will make the above operation justifiable, and its result correct. However, when we write down

$$\int_0^{4\pi} \sqrt{1-\cos x}\,dx$$

we tacitly assume that everybody knows that by $\sqrt{1-\cos x}$ we mean the non-negative square root, and with *that* definition (1) is incorrect.

In a parametric representation $x=\phi(t)$, $y=\psi(t)$ it is necessary to define unambiguously not only the functions $\phi(t)$ and $\psi(t)$, but also the region over which t varies. For example, if t is assumed to run through *all real values*, the equations

$$x = e^t, \qquad y = e^t \tag{2}$$

represent the portion of the line $x=y$ in the first quadrant. If t varies over *positive real values* only, then our equations represent only that portion of the line which lies to the right of the point (1, 1); and the equations (2) represent the entire line $x=y$ (with the exception of the origin) if t is allowed to assume complex values, and its region of variability in the complex t plane consists of two lines, the real axis and a line parallel to it at the distance π.

Often the tacit assumption will be that t runs over all real values, but this convention should be made clear to the students, and one should also explain that it is not always appropriate. Take another one of Mr. Fleshler's examples,

$$x = \sin t, \qquad y = \sin t. \tag{3}$$

These equations can be made to represent the entire line $x=y$ by letting t vary in the complex plane over a broken line which joins the points $(-\pi/2)-i\infty$, $(-\pi/2)$, $(\pi/2)$, $(\pi/2)+i\infty$ in this order. If one lets t vary over all real values, one obtains *not* so much *the segment* joining $(-1, -1)$ and (1, 1), *but* rather *a curve* which covers this segment infinitely often (and has an infinite

length!). If this is pointed out to a student familiar with the periodic property of trigonometric functions, he will see it, but on being challenged to specify a reasonable region of variation for t, his first guess is likely to be $0 \leqq t \leqq 2\pi$, which leads again not to the segment but to a closed curve covering this segment twice. Actually, one possible interval of definition is $-\pi/2 \leqq t \leqq \pi/2$. It is not the only one to lead to the desired result, but some such specification is necessary. In some case one will have to consider equations (3) in the interval $0 \leqq t \leqq \pi/2$, when they represent the segment joining the origin and the point (1, 1), and so on.

By a discussion of (3) in various (real) ranges of t the student is introduced at an early stage to the necessity of a precise definition, and the ground is prepared for the demand of the more advanced theory that a correct definition of a function should always include (a) the region of variability of the independent variable, and (b) an unambiguous rule for finding the value of the function for each admissible value of the independent variable.

AN ALTERNATE DERIVATION OF A WELL-KNOWN INTEGRATION FORMULA

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The derivation of the formula for the indefinite integral of the trigonometric tangent function commonly given in classrooms and in textbooks is the following:

$$\int \tan x dx = \int \frac{\sin x}{\cos x} dx = \int \frac{-d(\cos x)}{\cos x} = -\int \frac{d(\cos x)}{\cos x} \tag{1}$$
$$= -\ln \cos x + C = \ln \sec x + C.$$

This derivation, including the integration, is based on four formulas, from trigonometry and calculus, namely,

$$\tan x = \frac{\sin x}{\cos x}, \tag{2}$$

$$d(\cos x) = -\sin x dx, \tag{3}$$

$$\int a dv = a \int dv, \tag{4}$$

$$\int \frac{dv}{v} = \ln v + C. \tag{5}$$

The following derivation of the formula is perhaps more appealing because it is shorter and more direct:

$$\int \tan x dx = \int \frac{d(\sec x)}{\sec x} = \ln \sec x + C = -\ln \cos x + C. \tag{6}$$

This derivation, including the integration, is based on only two formulas, from

calculus, namely,

$$d(\sec x) = \sec x \tan x dx \tag{7}$$

and (5).

The alternate method of derivation applies, of course, also to the trigonometric cotangent function, and to the hyperbolic tangent and cotangent functions.

Generally, the differential equation

$$df(x) = f(x)g(x)dx, \tag{8}$$

of which (7) is a special case, and where $g(x)$ is an arbitrary function of x, has the solution

$$f(x) = C'e^{\int g(x)dx}, \tag{9}$$

so that for any integrable function $g(x)$ there is a function $f(x)$, given by (9), which satisfies (8) and, therefore,

$$\int g(x)dx = \ln f(x) + C. \tag{10}$$

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, Champlain College

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Champlain College, Plattsburg, New York. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTIONS

E 986. *Proposed by W. R. Ransom, Tufts College*

Show that there are fifty right triangles whose sides are integers less than 100.

E 987. *Proposed by W. L. Stamey and J. L. Zemmer, Jr., University of Missouri*

Prove that the interior of the strip, on a cartesian grid, bounded by two arbitrarily close parallel straight lines, contains either no points with integer coordinates or infinitely many such points.

E 988. *Proposed by Leo Moser, Texas Technological College*

Let a, b, x_0 be positive integers and let $x_1=ax_0+b, \cdots, x_n=ax_{n-1}+b$. Prove that x_n cannot be prime for all n.

E 989. *Proposed by C. W. Bruce, Tennessee Polytechnic Institute*

If the angle between the blades of scissors is less than the angle of friction of the metal on the cloth being cut, there will be a little tendency for the cloth to move forward as the scissors are being closed. How can a pair of scissors be constructed so that one blade is straight and there is a constant angle between the cutting edges?

E 990. *Proposed by Victor Thébault, Tennie, Sarthe, France*

If a sphere with center at the orthocenter of an orthocentric tetrahedron intersects the lines joining the midpoints of the edges of the tetrahedron, then the 24 distances of the vertices from the intersections on those lines joining the midpoints of the edges issuing from these vertices are all equal. (The analogous property for the triangle was announced by Steiner and proved by Droz-Farny. See Johnson, *Modern Geometry*, p. 256.)

SOLUTIONS

Arithmetic-Harmonic Mean

E 956 [1951, 189]. *Proposed by H. F. Sandham, Trinity College, Ireland*

Let a_0 and b_0 be positive numbers and define a_{n+1} and b_{n+1} as the arithmetic and harmonic means, respectively, of a_n and b_n. Show that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = (a_0b_0)^{1/2}.$$

Solution by Chandler Davis, University of Michigan. Since the case $a_0=b_0$ is trivial we assume $a_0>b_0$. Then $a_0>a_1>\cdots>b_1>b_0$, and the sequences $\{a_n\}$ and $\{b_n\}$ both converge. Call the limits a and b. From the expressions for the arithmetic and harmonic means we find

$$a_{n+1} - b_{n+1} = (a_n - b_n)^2/2(a_n + b_n) < (a_n - b_n)/2,$$

so that $a_n-b_n\to 0$, or $a=b$. Similarly we find that $a_{n+1}b_{n+1}=a_nb_n$, or

$$a^2 = \lim a_nb_n = a_0b_0,$$

and the proof is complete.

Also solved by F. Bagemihl and W. Seidel (jointly), L. F. Boron, A. L. Epstein, E. Franz, V. C. Harris, R. E. Horton, R. Huck, H. A. James, P. G. Kirmser, M. S. Klamkin, H. C. Kranzer, A. E. Livingston, Jerome Manheim, Burnett Meyer, Prasert Na Nagara, S. T. Parker, L. B. Rall, L. A. Ringenberg, Azriel Rosenfeld, E. C. Smith, M. R. Spiegel, O. E. Stanaitis, Elijah Swift, P. M. Treuenfels, F. Underwood, J. H. Wahab, Albert Wilansky, R. E. Wild, F. H. Young, and the proposer.

Palindromes

E 957 [1951, 189]. *Proposed by C. O. Oakley, Haverford College*

A *palindrome* is an integer that reads the same forwards as backwards as, for example, 2332. By definitions the ten digits 0, 1, 2, $\cdots$, 9 will be considered as palindromes. How many palindromes are there of at most n digits?

Solution by H. C. Kranzer, New York University. A palindrome of an even number, $2k$, or an odd number, $2k-1$, of digits is uniquely determined by its first k digits, which are arbitrary except that the first digit may not be zero (with the single trivial exception of 0 itself). Thus, if we let $f(p)$ be the number of non-zero palindromes of exactly p digits, we have, for $k \geqq 1$,

$$f(2k-1) = f(2k) = 9 \cdot 10^{k-1}.$$

We wish to find

$$F(n) = 1 + \sum_{p=1}^{n} f(p).$$

Let n be even. Then

$$F(n) = 1 + 2\sum_{k=1}^{n/2} f(2k) = 1 + 18\sum_{k=1}^{n/2} 10^{k-1} = 2 \cdot 10^{n/2} - 1.$$

Let n be odd. Then

$$F(n) = F(n-1) + f(n) = 11 \cdot 10^{(n-1)/2} - 1.$$

Also solved by R. V. Andree, D. R. Clutterham, Carl Cohen, R. E. Horton, H. A. James, M. S. Klamkin, G. R. Mach, Jerome Manheim, Prasert Na Nagara, J. D. Neff, C. S. Ogilvy, Bart Park, S. T. Parker, Azriel Rosenfeld, J. J. Rowland, E. P. Starke, C. W. Trigg, and the proposer.

An Interesting Determinant

E 958 [1951, 189]. *Proposed by Herbert Scarf, Temple University*

If $a_n = 1/n!$ show that

$$\begin{vmatrix} a_1 & a_0 & 0 & 0 & \cdots & 0 \\ a_2 & a_1 & a_0 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & a_0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ a_n & \cdot & \cdot & \cdot & \cdots & a_1 \end{vmatrix} = a_n.$$

I. *Solution by J. H. Wahab, University of North Carolina.* The result is obviously true for determinants of orders 1 and 2. If the result is assumed for all orders less than n, then expansion by cofactors of the elements of the first rows yields for the determinant D_n of order n:

$$D_n = a_{n-1} - a_2a_{n-2} + a_3a_{n-3} - \cdots (-1)^{n+1}a_n$$

$$= (1/n!)\left[\binom{n}{1} - \binom{n}{2} + \binom{n}{3} - \cdots (-1)^{n+1}\right] = 1/n! = a_n.$$

II. *Solution by F. H. Young, University of Oregon.* Subtract the first column from the second; in the resulting determinant subtract twice the second column from the third; in the resulting determinant subtract three times the third column from the fourth; and continue. We may show by induction that when this process is completed the resulting determinant is triangular with the jth element along the principal diagonal equal to $1/j$, from which the result follows.

III. *Solution by R. G. Buschman, University of Colorado.* Let $f(x) = 1/p(x)$, where $p(x) = \Sigma_{k=0}^{\infty} a_k x^k$, $a_0 \neq 0$. Then the power series expansion for $f(x)$ is $\Sigma_{k=0}^{\infty} b_k x^k$, where

$$b_n = (-1)^n D_n / b_0^{n+1},$$

D_n being the determinant of the problem. Therefore

$$D_n = (-1)^n b_n b_0^{n+1}.$$

Now consider the special case $a_n = 1/n!$. Then $p(x) = e^x$, $f(x) = e^{-x}$, and $b_n = (-1)^n/n!$. Hence $D_n = 1/n! = a_n$.

Also solved by A. C. Aitken, P. M. Anselone, F. Bagemihl, H. D. Block, Ralph Calvert, Chandler Davis, J. C. Eaves, C. V. Fronabarger, W. W. Funkenbusch, E. I. Gale, Louisa Grinstein, E. Ikenberry and W. A. Rutledge (jointly), H. A. James, R. S. Kingsbury, M. S. Klamkin, Violet Larney, S. T. Parker, F. A. E. Pirani, Azriel Rosenfeld, M. R. Spiegel, O. E. Stanaitis, Elijah Swift, F. Underwood, Louise Wolf, and the proposer.

A Special Kasner Oval

E 959 [1951, 189]. *Proposed by C. S. Ogilvy, Columbia University*

Let C_0 be a square, C_{n+1} the convex polygon obtained from C_n by measuring off one fourth of the length of every side of C_n and cutting off the corners, and $C = \lim_{n\to\infty} C_n$. Prove that C consists of four parabolic arcs.

Solution by Chandler Davis, University of Michigan. Locate C_0 with vertices at (1, 0), (0, 1), (−1, 0), (0, −1). We will show that C consists of four arcs of parabolas with axes parallel to the coordinate axes. By symmetry we need consider only the half of the first quadrant lying above $y = x$.

Now it is clear that the midpoints of the sides of C_n are midpoints of sides of C_{n+1}, C_{n+2}, $\cdots$, and therefore are on C. Also they are dense on C. On C_1, the points (1/2, 1/2) and (0, 3/4) are such midpoints. At those points C_1 is tangent to the parabola $y = 3/4 - x^2$. To finish the solution it is enough to note the following known property of a parabola: If the tangents to a parabola at any

two points A and B on the parabola meet in T, and if P is the midpoint of AT and Q the midpoint of BT, then PQ is tangent to the parabola at the midpoint of PQ.

Also solved by H. A. James, H. C. Kranzer, J. H. Wahab, and the proposer.

The proposer pointed out the following definition of a *Kasner oval.* Let P_0 be a plane convex polygon, and let P_{n+1} be the convex polygon obtained from P_n by measuring off $(1/r)$th of the length of every side of P_n from each vertex, and cutting off the corners. Then $\lim_{n\to\infty} P_n$ is called a *Kasner oval.* Of course, that the successive polygons remain convex and Jordan, places a restriction on the range of r.

A Chain of Circles in a Circle

E 960 [1951, 189]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Let AB be a diameter of a circle (O) and let C and D be the midpoints of the radii OA, OB. Let the circles $A(O)$, $B(O)$ cut (O) in A_1, B_1 on the same side of AB. On the other side of AB draw the semicircles $C(O)$ and $D(O)$. Consider the sequence of circles (O_1), (O_2), (O_3), $\cdots$, (O_n), $\cdots$, tangent to the arcs OA_1, OB_1 of $A(O)$ and $B(O)$ and, in turn, to (O), (O_1), (O_2), $\cdots$, (O_{n-1}), $\cdots$. Also consider the sequence of circles (O_1'), (O_2'), (O_3'), $\cdots$, (O_n'), $\cdots$ tangent to the semicircles $C(O)$ and $D(O)$ and, in turn, to (O), (O_1'), (O_2'), $\cdots$, (O_{n-1}'), $\cdots$. Show that the sum of the lengths of the circumferences of all the circles (O_n) and (O_n') is equal to the length of the circumference of (O).

Solution by C. S. Ogilvy, Columbia University. From the symmetry it is obvious that all the circles (O_n) and (O_n') form a row of externally tangent circles, with centers on a diameter of (O), and whose end members are tangent to (O). Thus if C_n is the circumference and D_n the diameter of circle (O_n), and similarly for the primed sequence, we have

$$\sum_{n=1}^{\infty} C_n + \sum_{n=1}^{\infty} C_n' = \pi\left(\sum_{n=1}^{\infty} D_n + \sum_{n=1}^{\infty} D_n'\right) = \pi(AB).$$

Of course any row whatever of circles ranged in this way along a diameter has the same property.

Also solved by H. A. James, M. S. Klamkin, Joseph Langr, and the proposer.

If r_n, r_n' are the radii of circles (O_n), (O_n'), (O), then it is easily shown that

$$r_n = r/2n(n+1), \qquad r_n' = r/(4n^2-1),$$

and that $\Sigma r_n = \Sigma r_n' = r/2$.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general problems in well known text books or results found in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4458. *Proposed by Z. A. Melzak, McGill University, Montreal*

Let $f(x)$ be a polynomial of degree n. Then

$$f(x+y) = \frac{y(y+1)(y+2)\cdots(y+n)}{n!}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{f(x-k)}{y+k}.$$

4459. *Proposed by D. J. Newman, New York University*

Find an asymptototic expression for the number of integers, not exceeding x, each of which has the property that each of its prime divisors divides it to the second power at least.

4460. *Proposed by Paul Erdös, University of Aberdeen, Scotland*

Let $f(1)=1$,

$$f(n) = (1+c_1)f(n-1) - c_2f(n-2) - \cdots - c_{n-1}f(1),$$

$c_i \geqq 0$, $i=1, 2, \cdots$, $c_1=c_2+c_3+\cdots$. What is the necessary and sufficient condition for the convergence of $f(n)$?

4461. *Proposed by George Grossman, De Witt Clinton High School, New York City*

A die is made of homogenous material in the shape of a rectangular parallelepiped and is tossed onto a level surface from which it will not bounce. What is the probability that a particular face will come up on top?

4462. *Proposed by M. R. Spiegel, Rensselaer Polytechnic Institute, Troy, New York*

If

$$S_{2m} = \frac{1}{2^{2m}} + \frac{1}{4^{2m}} + \frac{1}{6^{2m}} + \cdots.$$

Find the value of

$$\frac{S_2}{2\cdot 3} + \frac{S_4}{4\cdot 5} + \frac{S_6}{6\cdot 7} + \cdots.$$

SOLUTIONS

A Determinant Related to the Permutation Matrix

4383 [1950, 120]. *Proposed by Robert Steinberg, University of California, Los Angeles*

Let n symbols be ordered in two different ways, and let a_{ij} denote the number of symbols common to the first i in the first ordering and the first j in the second ordering. Prove that the determinant $|a_{ij}|$ is 1 if the transition from the first ordering to the second is effected by an even permutation and -1 if it is effected by an odd permutation.

Solution by the Proposer. Let the symbols in the first ordering be numbered $1, 2, \cdots, n$, and in the second ordering $p_1, p_2, \cdots, p_n$. Then, after defining $a_{0j}=a_{i0}=0$ for convenience, one can readily verify the following equalities which are true for $i, j=1, 2, \cdots, n$:

If $i>p_j$, then $a_{i,j}-a_{i,j-1}=a_{i-1,j}-a_{i-1,j-1}=1$.

If $i<p_j$, then $a_{i,j}-a_{i,j-1}=a_{i-1,j}-a_{i-1,j-1}=0$.

If $i=p_j$, then $a_{i,j}-a_{i,j-1}=1$ and $a_{i-1,j}-a_{i-1,j-1}=0$.

Hence the expression $a_{i,j}-a_{i,j-1}-a_{i-1,j}+a_{i-1,j-1}$ is 1 if $i=p_j$ and 0 otherwise. Thus, if we subtract the $(n-1)$st row of the matrix (a_{ij}) from the nth, the $(n-2)$nd from the $(n-1)$st, $\cdots$, the 1st from the 2nd, and do the same to the columns, we obtain the matrix (δ_{ip_j}). This latter matrix is just the permutation matrix corresponding to the permutation $i\rightarrow p_i$, and has the determinant 1 or -1 according as the permutation is even or odd.

A compact expression for the relationship between (a_{ij}) and (δ_{ip_j}) can be obtained as follows. The operations performed on the rows of (a_{ij}) are equivalent to left multiplication by the matrix which has 1's in the main diagonal, -1's in the sub main diagonal, and 0's elsewhere. This matrix has as inverse the matrix M_{sub} which has 1's on and below the main diagonal and 0's elsewhere. A corresponding result holds for the columns of (a_{ij}): one need only replace the words left, sub, below by right, super, above. The final result is

$$(a_{ij}) = M_{\text{sub}}(\delta_{ip_j})M_{\text{sup}}.$$

Orthoptic Locus of Conic Inscribed in a Triangle

4386 [1950, 188]. *Proposed by Victor Thébault, Tennie Sarthe, France*

In any triangle the Monge circle (orthoptic locus) of the inscribed conic concentric with the circumcircle is orthogonal to the polar circle.

Solution by L. M. Kelly, Michigan State College. The hypothesis that the center of the conic be the circumcenter is superfluous. We will prove the theorem for any inscribed conic and its associated orthoptic locus, appealing to the following well established properties.

1. The tangents from a point P to a conic inscribed in a complete quadrilateral are corresponding lines in an involution. The lines joining the point P to the opposite ends of the diagonals are also pairs of corresponding

elements in this involution. (Desargues)

2. Circles on the diagonals of a complete quadrilateral (as diameters) are coaxial. (Gauss-Bodenmiller)
3. A circle orthogonal to two given circles is orthogonal to every circle coaxial with them.

Consider any conic inscribed in a triangle and draw a fourth tangent to the conic. The conic is now inscribed in a complete quadrilateral. The circles on the diagonals intersect in two points P and P', by (2). The tangents from P to the conic and the pairs of lines joining P to the opposite vertices of the quadrilateral are elements of an involutory pencil, by (1). Furthermore the pairs of rays joining P to the opposite ends of a diagonal are at right angles. Thus the involution is circular and the tangents from P to the conic are perpendicular. Similarly for P'. Thus the orthoptic locus passes through P and P' and being in general a circle (in the case of a parabola, a line) is coaxial with the three circles on the diagonals. Each of these circles passes through a vertex and corresponding altitude foot of the original triangle. Thus each is orthogonal to the polar circle (Johnson, *Modern Geometry*, p. 177). By (3) the polar circle is orthogonal to the orthoptic locus as required.

Also solved by A. E. Landry and the proposer.

Sums of rth Powers of Divisors

4387 [1950, 188]. *Proposed by Paul Bateman, University of Illinois*

If $\sigma_r(n)$ denotes the sum of the rth powers of the divisors of the positive integer n, prove that

$$\sigma_r(n)\sigma_r(m) = \sum_{d\mid(n,m)} d^r\sigma_r(nm/d^2),$$

where d runs through all the common divisors of n and m.

Solution by the Proposer. Since for $(n, m)=1$ it is clear that $\sigma_r(nm)=\sigma_r(n)\sigma_r(m)$, it suffices to prove the formula in the case where n and m are both powers of the same prime p. Suppose $n=p^a$, $m=p^b$, $0<a\leqq b$. Then

$$\begin{aligned}\sigma_r(p^a)\sigma_r(p^b) &= (1+p^r+p^{2r}+\cdots+p^{ar})(1+p^r+\cdots+p^{br})\\ &= 1+2p^r+3p^{2r}+\cdots+(a+1)p^{ar}+(a+1)p^{(a+1)r}+\cdots\\ &\quad+(a+1)p^{br}+ap^{(b+1)r}+\cdots\\ &\quad+3p^{(a+b-2)r}+2p^{(a+b-1)r}+p^{(a+b)r}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\sum_{d\mid(p^a,p^b)} d^r\sigma_r\left(\frac{p^ap^b}{d^2}\right) &= \sum_{\lambda=0}^{a} p^{\lambda r}\sigma_r(p^{a+b-2\lambda})\\ &= \sum_{\lambda=0}^{a} p^{\lambda r}(1+p^r+\cdots+p^{(a+b-2\lambda)r}).\end{aligned}$$

It is easily seen that these two expressions are the same.

Also solved by T. M. Apostol, Emilie V. Haynsworth, Roger Lessard, and N. T. Seeley, Jr.

Editorial Note. Apostol proves the problem as a special case of the following theorem: *Let $f(n)$ be a multiplicative function, $f(1)=1$, and let $g(n)$ be a completely multiplicative function, $g(1)=1$. Then the equation*

$$f(m)f(n) = \sum_{d|(m,n)} g(d)f(mn/d^2) \tag{1}$$

is equivalent to the equation

$$f(p)f(p^a) = g(p)f(p^{a-1}) + f(p^{a+1}), \qquad p = \text{prime}, \ a \geqq 1, \tag{2}$$

and also to the equation

$$\sum_{n=1}^{\infty} f(n)n^{-s} = \prod_{p} \frac{1}{1 - f(p)p^{-s} + g(p)p^{-2s}} \tag{3}$$

whenever the Dirichlet series is absolutely convergent. The product is extended over all primes.

Since equation (2) is easily verified for $f(n)=\sigma_r(n)$ with $g(n)=n^r$, this theorem solves the proposed problem. Equation (3) also solves the problem since, for the same choice of $f(n)$ and $g(n)$, both sides are the same as $\zeta(s)\zeta(s-r)$, where $\zeta(s)$ is the Riemann zeta-function.

Components of the Set $|(z-z_1)\cdots(z-z_n)| \leqq 1$

4388 [1950, 188]. *Proposed by Paul Erdös, University of Aberdeen, and W. H. Fuchs, Cornell University*

Let

$$f(z) = \prod_{i=1}^{n} (z - z_i), \qquad |z_i| \leqq 1.$$

Consider the set $|f(z)| \leqq 1$. Prove that it consists of at most $n-1$ components.

Solution by G. Szegö, Stanford University. A confluence of the components of the lemniscate $|f(z)| = \text{constant}$ takes place whenever $f'(\zeta)=0$. Let $\zeta_1, \zeta_2, \cdots, \zeta_{n-1}$ be the roots of this equation; we have to show that $\min |f(\zeta_i)| \leqq 1$. But

$$\begin{aligned} \min |f(\zeta_i)|^{n-1} &\leqq |f(\zeta_1)|\,|f(\zeta_2)| \cdots |f(\zeta_{n-1})| \\ &= n^{-n}|f'(z_1)| \cdots |f'(z_n)| = n^{-n}\prod_{i\neq k}|z_i - z_k|. \end{aligned}$$

The maximum of this product is attained for $f(z)=z^n-1$ (see I. Schur, *Mathematische Zeitschrift*, 1918, p. 385) in which case

$$|f'(z_1)| = \cdots = |f'(z_n)| = n.$$

Hence $\min |f(\zeta_i)| \leqq 1$.

Also solved by the proposers.

Simultaneous Linear Equations

4389 [1950, 188]. *Proposed by F. J. Dyson, Institute for Advanced Study*

Given N numbers a_m satisfying the N equations

$$\sum_{m=1}^{N} \frac{a_m}{m+n} = \frac{4}{2n+1}, \qquad n = 1, 2, \cdots, N,$$

prove that

$$\sum_{m=1}^{N} \frac{a_m}{2m+1} = 1 - \frac{1}{(2N+1)^2}.$$

This problem arose in an investigation of the diffraction of light (H. Levine and J. Schwinger, *Physical Review*, 1948, p. 970).

I. *Solution by Ernest Trost, Technikum Winterthur, Switzerland.* We can prove the following more general theorem: *If the N numbers a_m satisfy the N equations*

$$\sum_{m=1}^{N} \frac{a_m}{A_m + B_n} = \frac{-a_{N+1}}{A_{N+1} + B_n}, \qquad (n = 1, 2, \cdots, N),\ A_m + B_n \neq 0,$$

we have

$$\sum_{m=1}^{N} \frac{a_m}{A_m + x} = \frac{a_{N+1}}{A_{N+1} + x}\left\{\prod_{k=1}^{N} \frac{(x - B_k)(A_{N+1} - A_k)}{(x + A_k)(A_{N+1} + B_k)} - 1\right\}.$$

Consider the polynomial of the Nth degree in x

$$f(x) = \prod_{k=1}^{N+1}(x + A_k) \sum_{m=1}^{N+1} \frac{a_m}{x + A_m}. \tag{1}$$

$f(x)$ vanishes for $x = B_n$ $(n = 1, 2, \cdots, N)$. Hence we have

$$f(x) = C \cdot \prod_{i=1}^{N}(x - B_i), \tag{2}$$

where C denotes a constant. Putting $x = -A_{N+1}$ we obtain

$$a_{N+1} \cdot \prod_{k=1}^{N}(A_k - A_{N+1}) = (-1)^N \cdot C \cdot \prod_{k=1}^{N}(A_{N+1} + B_k),$$

and thus

$$C = a_{N+1} \prod_{k=1}^{N} \frac{(A_{N+1} - A_k)}{(A_{N+1} + B_k)}.$$

From (1) and (2) we deduce

$$\sum_{m=1}^{N+1} \frac{a_m}{x+A_m} = \frac{a_{N+1}}{x+A_{N+1}} \prod_{k=1}^{N} \frac{(x-B_k)(A_{N+1}-A_k)}{(x+A_k)(A_{N+1}+B_k)},$$

which proves the theorem.

Putting $A_m=m$, $B_n=n$ $(m, n=1, 2, \cdots, N)$, $A_{N+1}=x=\frac{1}{2}$, $a_{N+1}=-2$, we have the special case of the problem.

II. *Solution by G. Szegö, Stanford University.* For every polynomial $\phi(x)$ of degree $N-1$:

$$\phi(x) = \sum_{m=1}^{N} \frac{\phi(-m)}{f'(-m)} \frac{f(x)}{x+m}$$

identically. Take, in particular,

$$\phi(x) = \frac{2}{f(\frac{1}{2})} \frac{f(\frac{1}{2})f(x) - f(-\frac{1}{2})f(-x)}{x+\frac{1}{2}}$$

so that

$$\phi(n) = \frac{2}{n+\frac{1}{2}} f(n), \qquad n = 1, 2, \cdots, N,$$

$$\frac{2}{n+\frac{1}{2}} = \sum_{m=1}^{N} \frac{\phi(-m)}{f'(-m)} \frac{1}{m+n}.$$

Consequently

$$a_m = \frac{\phi(-m)}{f'(-m)} = \frac{4(-1)^{m-1}(N+m)!}{(2m-1)(2N+1)(N-m)!m!(m-1)!}.$$

Now

$$\sum_{m=1}^{N} \frac{a_m}{2m+1} = \sum_{m=1}^{N} \frac{\phi(-m)}{f'(-m)} \frac{1}{2m+1} = \frac{1}{2} \frac{\phi(\frac{1}{2})}{f(\frac{1}{2})}$$

$$= \frac{(f(\frac{1}{2}))^2 - (f(-\frac{1}{2}))^2}{(f(\frac{1}{2}))^2} = 1 - \left(\frac{f(-\frac{1}{2})}{f(\frac{1}{2})}\right)^2 = 1 - \frac{1}{(2N+1)^2}.$$

Also solved by N. T. Seeley, Jr., O. E. Stanaitis, and G. Szegö (two additional solutions).

Equilateral Hyperbola Inscribed in a Triangle

4393 [1950, 265]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Show that: (1) the center ω of an equilateral hyperbola $(\mathfrak{H})$ inscribed in a triangle ABC, the orthocenter H of this triangle, and the center N of the nine-point circle (N) of the triangle of contact $\alpha\beta\gamma$ of $(\mathfrak{H})$ with the sides of ABC are collinear. (2) The polar circle of triangle ABC is tangent to the circle (N).

(3) If BC, CA, AB; $\beta\gamma$, $\gamma\alpha$, $\alpha\beta$ respectively intersect one of the asymptotes of $(\mathfrak{C})$ in A', B', C'; α', β', γ', we have

$$\omega N/\omega H = \omega\alpha' \cdot \omega\beta' \cdot \omega\gamma'/\omega A' \cdot \omega B' \cdot \omega C'.$$

Solution by Roscoe Woods, State University of Iowa. Without loss of generality, the equation of $(\mathfrak{C})$ may be taken as $xy=1$. Its center ω is then the origin O of coördinates. By the "point t" on $(\mathfrak{C})$ we shall mean the point whose coordinates are (t, t^{-1}). Let t_1, t_2, t_3 be the points α, β, γ of contact of $(\mathfrak{C})$ with the sides of triangle ABC. The equations of the tangents to $(\mathfrak{C})$ at α, β, and γ are then $x+t_i^2y=2t_i$, $i=1, 2, 3$. Then, if C is the point of intersection of the tangents at α and β, its coördinates are

$$\left(\frac{2t_1t_2}{t_1+t_2}, \frac{2}{t_1+t_2}\right).$$

The coördinates of A and B may be found by cyclic permutation of the subscripts.

For convenience put $S_1=t_1+t_2+t_3$, $S_2=t_2t_3+t_3t_1+t_1t_2$, $S_3=t_1t_2t_3$. It is then found that the coördinates of H and N are respectively

$$\left(\frac{-2+2S_1S_3}{S_1S_2-S_3}, \frac{2S_2-2S_3^2}{S_1S_2-S_3}\right), \quad \left(\frac{-1+S_1S_3}{4S_3}, \frac{S_2-S_3^2}{4S_3}\right).$$

Since the slopes of OH and ON are equal, property (1) is proved.

To demonstrate (2), use is made of the following well known facts: (a) the center of an equilateral hyperbola circumscribing a triangle lies on the nine-point circle of the triangle, (b) the center of the polar circle of a triangle coincides with the orthocenter, and (c) the square of the radius of the polar circle is the product of the segments in which the orthocenter divides an altitude. The demonstration is completed by a short calculation which shows that the square of the radius of the polar circle of ABC is $\overline{OH}^2$.

The squares of the distances ON and OH are readily found to be $T^2/16S_3^2$ and $4T^2/(S_1S_2-S_3)^2$, respectively where $T^2=(1+t_1^2t_2^2)\ (1+t_2^2t_3^2)(1+t_3^2t_1^2)$. The products of the intercepts of the sides of the triangles $\alpha\beta\gamma$ and ABC on the x-axis (one of the asymptotes of $(\mathfrak{C})$) are, respectively, $S_1S_2-S_3$ and $8S_3$. Thus property (3) is established.

Also solved by E. D. Camier, Joseph Langr, and the Proposer.

Polynomials Related to Stirling Polynomials

4396 [1950, 342]. *Proposed by D. H. Browne, Buffalo, New York*

With the notation

$$P(s, t) = \prod_{m=0}^{s} (1 + mt),$$

let

$$S(n, t) = P(n, t)\sum_{k=0}^{n} k!t^k/P(k, t).$$

Show that for $n \geqq 1$

$$S(n, t) \equiv P(n-1, t) \pmod{n+1},$$

and that for $n > 2$, $t \leqq n$,

$$S(n, t) \equiv 0 \pmod{t+1}.$$

Solution by N. T. Seely, Jr., Agricultural, Mechanical and Normal College, Pine Bluff, Arkansas. From the hypothesis we have

$$\begin{aligned}
&S(n, t) - P(n-1, t) \\
&= P(n, t)\sum_{k=0}^{n} \frac{k!t^k}{P(k, t)} - P(n-1, t) \\
&= P(n, t) - P(n-1, t) + \sum_{k=1}^{n} k!t^k \frac{P(n, t)}{P(k, t)} \\
&= ntP(n-1, t) + \sum_{k=1}^{n-1} k!t^k \left\{\frac{P(n-1, t)}{P(k-1, t)} + (n-k)t\frac{P(n-1, t)}{P(k, t)}\right\} + n!t^n \\
&= (n+1)t\frac{P(n-1, t)}{P(0, t)} + (n+1)\sum_{k=1}^{n-2} k!t^{k+1}\frac{P(n-1, t)}{P(k, t)} + (n+1)(n-1)!t^n.
\end{aligned}$$

(The last summation is obtained from the preceding one by combining the second term of each summand with the first term of the next.) Thus,

$$S(n, t) - P(n-1, t) = (n+1)tS(n-1, t)$$

which is divisible by $n+1$, $n \geqq 1$.

Since $S(n, t)$ is a polynomial in t, the remainder on division by $t+1$, that is, $S(n, -1)$, must be shown to be divisible by $t+1$ for $n > 2$, $t \leqq n$. Thus

$$\begin{aligned}
S(n, -1) &= \sum_{k=0}^{n} (-1)^k k!\frac{P(n, -1)}{P(k, -1)} = \sum_{k=0}^{n-1} (-1)^k k! \sum_{m=k+1}^{n} (1-m) + (-1)^n n! \\
&= \sum_{k=0}^{n-1} (-1)^n k!k(k+1)\cdots(n-1) + (-1)^n n! \\
&= (-1)^n(n-1)!\sum_{k=0}^{n-1} k + (-1)^n n! \\
&= (-1)^n(n-1)!\frac{n}{2}(n-1) + (-1)^n n! \\
&= (-1)^n(n+1)!/2,
\end{aligned}$$

which is divisible by $t+1$ for $n>2$, $t\leqq n$.

Also solved by the Proposer.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 80 Waterman Street, Providence 6, Rhode Island, and not to any of the other editors or officers of the Association.

The Arithmetic Theory of Quadratic Forms. By B. W. Jones. Carus Mathematical Monograph No. 10. The Mathematical Association of America. (Distributed by John Wiley and Sons, Inc., New York.) 1950. x+212 pages. $3.00.

Let f and g be quadratic forms in n and m variables ($n\geqq m\geqq 1$), with coefficient matrices A and B respectively. The monograph deals principally with the conditions under which the form f represents the form g, or equivalently the existence of a matrix, X such that $X^TAX=B$. A and B have elements in a given field or ring, and X is to have elements in an assigned field or ring Starting with A, B, and X all having elements in the real field, the successive chapters narrow the fields or rings until finally the classic case where all elements are rational integers is treated.

A more detailed idea of the contents of the monograph is given by the chapter titles and principal topics in each chapter. I. Forms with real coefficients. Congruent forms. Canonical forms. Representation conditions. Automorphs and number of representations. II. Forms with p-atic coefficients. Definition and properties of p-atic numbers. Relation between rational representation modulo p^t, and p-atic representation. Hilbert and Hasse symbols. Application to representation problems. Universal forms. III. Forms with rational coefficients. Equivalence and reduction. Fundamental theorems on zero forms, and on rational congruences. Representation of one form by another. IV. Forms with coefficients in $R(p)$. Equivalence. Canonical forms. Numbers represented by forms. Conditions for equivalence. Forms with given invariants. Zero and universal forms. Automorphs. Binary forms. V. Genera and semi-equivalence. Definitions. Representation without essential denominator. Forms with integral coefficients and given invariants. VI. Representations by forms. Siegel's representation function. Asymptotic results. A representation function. VII. Binary forms. Automorphs. Number of representations. Quadratic ideals. Correspondence between ideal classes and classes of forms. Composition of classes. Genera. Reduced, and zero, definite forms. Reduced indefinite forms. Number

of classes. VIII. Ternary forms. Numbers represented by ternary genera. Number of representations. There is also a short bibliography of more recent work and an excellent set of problems, which will enable the reader to test his grasp in the material treated.

As has already been pointed out by Kloosterman, some of the proofs in Ch. II are incomplete. The essential difficulty is the passage from an infinite sequence of matrices, S_t, in $R(p)$, which satisfy $S_t^T A S_t \equiv B$ modulo p^t for each integer $t \geqq t_0$, to a matrix X satisfying $X^T A X = B$. Minor misprints were noted on pages 17 and 22.

The monograph is essentially self contained. It is presupposed only that the reader knows the fundamentals of matric theory and number theory. The style is clear, the presentation compact. More difficult concepts are clarified by well chosen examples. Starting with the basic ideas, in less than 200 pages we are led to the most recent developments in the subject, which heretofore have not been available in book form. Professor Jones' monograph is warmly recommended as an introduction to the arithmetic theory of quadratic forms.

J. D. ELDER

1. *Plane Trigonometry*. J. J. Corliss and W. V. Berglund. Houghton Mifflin Co. 1950, 12+388 pages. $3.00.
2. *Plane Trigonometry with Tables*. Gordon Fuller. McGraw-Hill, 1950, 11+270 pages. $2.75.
3. *College Trigonometry*. W. L. Hart, Heath and Co. 1951, 8+230+130 pages. $3.50.
4. *Plane Trigonometry—Alternate Edition with Tables*. E. R. Heineman. McGraw-Hill, 1950, 14+184+75 pages. $2.50.
5. *Trigonometry with Tables*. C. T. Holmes. McGraw-Hill, 1951, 9+246 pages. $3.00.
6. *Essentials of Plane Trigonometry*. J. B. Rosenbach, E. A. Whitman, and David Moskovitz. Ginn and Co. 1950, 8+168+15 pages.
7. *Plane and Spherical Trigonometry with Tables*. J. Shibli. (Third Edition), Ginn and Co. 1949, 12+262+94 pages.

All of these textbooks appear to be written carefully and should prove satisfactory in the trigonometry courses for which they are respectively intended. (3), (5) and (7) cover at least the fundamentals of spherical trigonometry with applications, while the others as their titles indicate cover only plane trigonometry. Except for (6) they all include the basic tables needed in trigonometry and all include the answers to at least one-half of the problems.

Except for (7) they all begin with the introduction of a coordinate system in the plane and the definition of the trigonometric functions of the general angle. (7) begins at a more elementary level with a discussion of measurement,

some historical remarks, a review of a few relevant theorems from high school plane geometry, and the definition of the trigonometric functions of an acute angle. (5) derives the formula $d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$ and applies it in two different coordinate systems to give an analytic proof of $\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$ for all α, β. (6) proves the addition formula for α, β acute by expressing the area of a triangle in two ways, while the other books prove the addition formula in the standard way first for acute α, β and then for all angles.

Except for (6) they all derive the properties of logarithms from scratch before taking up the logarithmic solution of triangles. The authors of (6) evidently assume that many students have had a previous exposure to logarithms, but they do include a final chapter VII on logarithms for students with inadequate preparation. All except (6) include a chapter on complex numbers including De Moivre's theorem. (5) includes a section on the series representation of functions and one on Euler's formula as well, while (7) includes without proof the Maclaurin expansions of $\sin x$, $\cos x$, e^x, Euler's formula and the definitions of the hyperbolic functions.

(1) and (3) devote some space to the general concepts of "variable" and "function" while (5) gives a brief explanation of "function." The other authors limit their discussion to trigonometric functions without explaining what a function is.

(1) and (5) introduce polar coordinates explicitly, while (2), (3), (4) and (7) discuss the polar form of a complex number. All the books discuss vectors while all except (2), (4) and (5) discuss projections. On the whole (1) and (3) seem strongest on applications of plane trigonometry while (5) seems weakest. All except (6) devote a fair amount of space to significant figures and the accuracy of results. Apparently (6) has only a few footnotes on the subject. All the books have a reasonable number of illustrative examples and problems for the student.

I shall end the review by mentioning a few additional features of each of the texts.

The authors of (1) are careful to emphasize concepts and topics such as functions, variables and polar coordinates which are important in other mathematics courses. There is a good discussion of computation with approximate numbers and of interpolation. There is a good chapter on applications. The reviewer was surprised to notice a slip on page 206 where the authors do not justify the disappearance of $\pm$. There are four tables, squares, 4 place trig and 5-place log and log trig.

(2) is a rather conservative trigonometry book including good sections on approximate numbers, significant figures and rounding off. There are 5-place log and log trig tables.

(3) is a comprehensive book on trigonometry for mature students. Especially noteworthy are the numerous problems, the comprehensive review of plane trigonometry, the appendices to plane trigonometry including ones on the logarithmic scale, logarithms of functions near 0° and 90°, plane surveying

and the use of reversible operations, the very thorough treatment of spherical trigonometry with the use of haversines, and the 17 tables including squares and square roots, 3-place log and log trig, 4-place and 5-place log, trig and log trig, auxiliary table for angles near 0° or 90°, natural logs, logs for compound interest, and haversines.

(4) like (2) is conservatively written. There are 4 tables, 4-place trig, 5-place log and log trig, and squares, cubes, square roots, cube roots and reciprocals.

(5) is especially good at relating trigonometry to analytic geometry and calculus. The proof for $\cos(\alpha-\beta)$, the definition of variables and functions, and the series representation of functions have been mentioned. There is a discussion of reversible operations in working with identities, common algebraic errors, the law of natural growth, e as a limit and the logarithmic and exponential functions as inverse functions. There are 3 tables, 5-place log, log trig and trig.

(6) is the shortest of the books and hence does not cover some of the material included in the others. It includes some historical notes, many illustrative examples with the analysis and computation in parallel columns, and a collection of general exercises at the end. In a pocket inside the back cover there are a ruler-protractor and a card with proportional parts for differences from 1 to 105 and a table of angles corresponding to certain rational values of the trigonometric functions.

As already pointed out (7) begins at an elementary level suitable for high school but still it covers as much material as most of the other texts by the end. It is very clear and spherical trigonometry with applications is covered quite well in 32 pages. There are 9 tables (many 5-place) including squares, square roots and reciprocals, natural logarithms, and hyperbolic functions.

L. J. BURTON

NEW BOOKS RECEIVED

Synthesis of Electronic Computing and Control Circuits, Annals 27. Harvard University Computation Laboratory. Cambridge, Harvard University Press, 1951. 278 pp.

Platon et la Recherche Mathematique de son epoque. Charles Mugler. Strasbourg. Editions. P. H. Heitz, 1948.

L'Acces Aux Principes de la Geometrie Euclidienne. By G. Bouligand. Paris, Librairie Vuibert, 1951, vii+87 pp. 320 fr.

General Homogeneous Coordinates in Space of Three Dimensions. By E. A. Maxwell. Cambridge University Press, 1951. xiv+169 pp. $2.75.

Programs for an Electronic Digital Computer. By Wilkes, Wheeler, Gill, Cambridge, Mass., Addison Wesley, 1951. 167 pp.

Mathematics of Statistics, Revised Edition. By J. F. Kenney and E. S. Keeping. New York, Van Nostrand. xiii+429 pp.

Mathematics for Technical Students, Book 2. By J. D. N. Gasson. Cambridge University Press, 1951. x+431 pp.

CLUBS AND ALLIED ACTIVITIES

EDITED BY L. F. OLLMANN, Hofstra College

Material for this department should be sent to Professor H. D. Larsen, Albion College, Albion, Michigan.

CLUB REPORTS, 1950–51

Sigma Phi Mu, Montclair, New Jersey State Teachers College

The papers presented to the membership of *Sigma Phi Mu*, the mathematics club at State Teachers College, Montclair, New Jersey, included the following:

The mathematics major in the elementary school, by Anna May Martin
Meteorology and mathematics, by William Lone
Mathematics and the slow student in high school, by Max A. Sobel
Probability, by Edward Molino.

The first meeting of the year was devoted to an initiation and welcoming of the Freshmen. Other social activities consisted of an annual Christmas Party with *Kappa Mu Epsilon*, a farewell party for the Seniors, and the annual May picnic.

A major project for the year was the initial publication of a club newspaper, *"The limit" as Sigma Pi Mu approaches the news*. This paper was written, edited and printed by members of the club. It contained scientific and feature articles, department and club news, editorials on current events in the department and the field of mathematics, a book review and various types of problems.

The officers for 1951–52 are: President, Joseph Sommer; Vice-President, Howard Mion; Secretary, Barbara Pearson; Treasurer, John Rowley.

Mathematics Club, University of Colorado

Included among the talks presented at the meetings of the *Mathematics Club* of the University of Colorado were:

Topics in topology, by A. L. Blakers
Meyers theorem and quadratic forms, by B. W. Jones
Consecutive powers and the form of binomial coefficients, by Paul Erdös
Infinite abelian groups, by K. A. Hirsch
The product of three linear forms, by Jurt Mahler
Mappings of spheres and disks, by A. Grudin
Point lattices and Farey sequences, by D. DeVol
Representations of primes, by N. C. Ankeny
Partitions and generating functions, by W. E. Briggs
Lattice points and hyperbolas, by B. Hunt
Operational calculus and applications to number theory, by J. R. Britton
Iteration of mappings of a metric space onto itself, by Albert Edrei
Crooked dice, by D. Hawkins.

The officers for the year were: President, B. Hunt; Secretary-Treasurer, R. G. Buschman.

Pi Mu Epsilon, University of California

The *California Alpha* chapter of *Pi Mu Epsilon* had ten meetings during the year, of which two were purely business meetings, two were initiation meetings held in private homes, and six were lecture meetings. The lectures were:

Theory of games, by Irving Glicksberg

Theory of braids, and a graphic approach to permutation groups, by James Jackson

Problem types in plasticity, by George Zizicus

Some results related to the fixed-point theorem—and stuff, by Dr. Robert Steinberg

Automatic computing machinery, by Dr. Harold Luxemberg

Transforms and tautochrones, by Dr. G. Milton Wing.

The Annual Calculus Prize examination was won by George S. Rasmussen and W. T. Root (tie).

Officers for 1951–52 are: Director, James Jackson; Vice-Director, Sharla Rita Perrine; Secretary, Mervin Muller; Faculty Adviser, Dr. Phil Hodge.

Mathematics Society, Massachusetts Institute of Technology

The *Mathematics Society* of Massachusetts Institute of Technology met weekly. The following were among the lectures given:

The existence of non-measurable sets on the sphere, by Prof. W. Hurewicz

Ergodic theory and probability, by Prof. W. Ambrose

Theory of elasticitity, by Prof. E. Reissner

Theory of characteristics and supersonic flow, by Prof. Lin

Asymptotic expansions, by Prof. Hildebrand

Fourier series, by Dr. Rudin

Geometry of differential equations, by Dr. Coddington

Maximum aspects of eigen problems in algebra and integral equations, by R. B. Davis

The Laplace transform, by W. Shields

Transcendental numbers, by H. Shapiro

Chess and mathematics, by H. Davis

Three proofs of Weierstrass' polynomial approximation theorem, by R. B. Kellogg

Klein's geometrical theory of algebraic equations, by W. Baily

Internal symmetric transforms, by R. Preisendorfer.

The Society published two issues of its *Bulletin* giving reviews of some of the lectures.

The officers for 1950–51 were: President, Bruce Kellogg; Vice-President, Rudolph Preisendorfer; Secretary-Treasurer, David Berkowitz; Bulletin Editor, Don Aronson.

Mathematics Club, Swarthmore College

The activities of the Swarthmore *Mathematics Club* during the past year

include lectures, problem contests, and participation in the William Lowell Putnam Mathematics contest.

The following talks were presented:

Characteristic properties of the circle, by R. A. Rosenbaum

Number partitions, by Arthur Mattuck

Number theory, by Oystein Ore

Properties of isogonal conjugates, by Louis Winer

Diophantine approximations, by H. W. Brinkmann

A mathematical model in the social sciences, by Lotte Lazarsfeld

Greek mathematics (in conjunction with the Classics Club), by Arnold Dresden

Some aspects of differential geometry in the large, by C. B. Allendoerfer

The prime number theorem, by Walter Leser

Continued fractions, by Joan Berkowitz.

Prizes for the contests sponsored by the club went to James VanderWeen, Arthur Mattuck, Louis Winer and Joan Berkowitz. Edward Prenowitz won the Putnam contest.

President-elect is Barbara Wolff.

Pi Mu Epsilon, University of Illinois

In addition to the usual business meetings, the Illinois *Alpha* chapter of *Pi Mu Epsilon*, University of Illinois, held two meetings during the school year 1950–51. They were:

A demonstration lecture on *The analog computer*, by Prof. A. T. Nordsieck and the Spring initiation meeting at which ninety-nine new members were taken into the fraternity. Prof. R. H. Bing of the University of Wisconsin addressed the meeting on *Some theorems in topology which are valid in two-space, but not in three-space*. Awards were given to Prof. J. W. Peters, G. E. Modesitt and Lloyd R. Welch.

The new officers for 1951–52 are: President, Thomas Elfe; Vice-President, John W. Toole; Secretary, Beverly Marshall; Treasurer, Richard Priest.

Zeno Club, Alfred University

The *Zeno Club* of Alfred University held eight regular meetings throughout the school year of 1950–51, and the following phases of mathematics were reported on:

Mathematics and world maps, by Dr. C. E. Rhodes

Equally likely, by Robert Lober

Arithmetic in the past, by Prof. V. Nevins

Magic squares, by Ralph Beals

Irrational numbers, by John Ging

Rank bi-serial coefficient of correlation, by Dr. Stephen Clark and Prof. John Freund

Projective number system, by Barbara Fischer

Poles and polars, by Dr. A. E. Whitford.

A dinner party was given by the faculty members and their wives for all mathematics majors and members of the Zeno Club.

In the spring the students in the club gave a picnic for the faculty and their families.

Officers of the club for the year were: President, Barbara Fischer; Vice-President, Robert Hultquist; Secretary-Treasurer, Jane Bette; Faculty Adviser, Dr. C. E. Rhodes.

Pi Mu Epsilon, Bucknell University

The following papers were presented to the *Pennsylvania Beta* chapter of *Pi Mu Epsilon* during the past year:

The background of necessary and sufficient, by Emil Polak

Diophantine problems, by Ralph Jones

Dimensional analysis, by Francis Huber

The cross ratio, by F. S. McFeely.

Officers elected for 1951–52 are: Director, Prof. W. K. Smith; Vice-Director, James E. Hole; Secretary, Rosemary Scheerer; Treasurer, C. Jerome Sechrist.

Kappa Mu Epsilon, Albion College

The *Michigan Alpha* chapter of *Kappa Mu Epsilon* at Albion College met monthly during 1950–51. The following programs were presented:

Arithmetic revisited, by Dr. H. D. Larsen

Wolfgang and Johann Bolyai, by Vinod Doshi

Gottfried Wilhelm Leibniz, by Robert Hooper

Neils Hendrick Abel, by James Young

Linkages, by Richard Burrows

A day of practice teaching, by Patricia Collins

Cryptography, by Milton Ehlert

The Laplace transform, by Albert Foster

The national convention of Kappa Mu Epsilon, by William Fryer

The planimeter, by William Horn.

The year's activities also included the presentation of the play, *It Can't Happen Here*, with the cast composed of initiates. The chapter also sponsored an open house in the Mathematics Department during Albion's annual "Meet The College" Day.

Officers elected for the year 1951–52 are: President, Paul Jones; Vice-President, William Ferguson; Secretary-Treasurer, Shirley Coffron; Faculty Sponsor, Dr. H. D. Larsen.

Kappa Mu Epsilon, Mount St. Scholastica College

The *Kansas Gamma* chapter of *Kappa Mu Epsilon* held bi-monthly meetings during the year 1950–51 following the chosen theme, *The development of mathematical algorithms*. The papers presented were:

Mathematics in the Orient, by Jeanne Culivan and Frances Donlon

Greek mathematics, by Ruth Link and Teresita Breitenbach
Hindu, Arabic and Persian developments, by Anne Robben and Jill Sullivan
Mathematics in the middle ages, by Margot Acree
Mathematics in the renaissance, by Elaine Barnes
Seventeenth century geometry, by Patti Shideler and Dorothy McManus
Development of the calculus, by Margaret McBride
The development of mathematics in the eighteenth century, by Mary Clare Brenwald.

A special feature in this year's program was the emphasis placed on visual aids for better understanding of principles involved.

The social program for the year consisted of a get-together picnic held for the pledges, the traditional English Wassail party at Christmas, and the formal initiation and banquet in the Spring. At the latter meeting four new members were initiated.

Special programs were the presentation of a skit *Mathematical relations* by the pledges and *It can't happen here* as a joint effort with the music society on the campus. A special contribution was made by Ruth Link in the form of an article, *Mysticism and mathematics*, which was published in the Mt. St. Scholastica quarterly magazine.

At the annual initiation Anne Robben was the recipient of the Hypatian award. This award is given each year for the oustanding contribution to the chapter. Miss Robben merited it because she presented a paper, *Some properties of the Simson line*, at the eighth biennial convention of the fraternity. Kathleen Feldhausen was presented with the underclassman award.

Officers elected to serve for 1951–52 are: President, Margaret McBride; Vice-President, Mary Brenwald; Secretary, Jill Sullivan; Treasurer, Anne Robben; Publicity, Elizabeth Loughlin; Musician, Kathleen Feldhausen. Sister Helen Sullivan will continue as Corresponding Secretary and Faculty Sponsor.

James G. White Mathematics Club, University of Kentucky

The following is a summary of the activities of the University of Kentucky *Mathematics Club* for 1950–51. The talks presented were:

The purpose of a mathematics club, by Dr. H. H. Downing
Mathematical squares, by Bill West
A story in mathematical terms, by Annette Siler
Mathematics for the million, by Gene Hagan
The mathematical aspects of weather, by Prof. Cordell Moore
Magic tricks, by Garland Dummitt
Topics on the early history of mathematics, by Dr. Aughtum Howard
Tisserand's Criterion, Gegenschein, and Trojan Asteroids, by Dr. Morris Davis.
Mathematics and logic, by Carl Faith.

The club joined with *Pi Mu Epsilon* in sponsoring a picnic for students and faculty.

The officers for the year are: President, Bill West; Vice-President, Roy Ellis; Secretary-Treasurer, Annette Siler; Faculty Advisor, Dr. James A. Ward.

Mathematics Club, Carleton College

The monthly meetings of the Carleton College *Mathematics Club* for the year 1950–51 included the following:

Theory of finite differences, by Mr. George Francis
The fourth dimension, by Norman Johnson
The mathematics of radiochemistry, by Dr. A. J. Miller
Introduction to spectral theory, by Mr. Winston Crum
Radiative transfer in stellar atmospheres, by A. F. Cook.

Dr. Warren Loud of the University of Minnesota spoke at the Annual banquet in January. His subject was *Curves of pursuit.*

The April meeting was conducted by the students with the theme *Mathematic for recreation.* Mathematical puzzles, charades, games, and a mathematics spelldown were included in the evening's program.

Officers elected for 1951–52 are: President, Gordon Grant; Vice-President-Treasurer, Norman Johnson; Secretary, Helen Heinzen; Faculty Adviser, Mr. Winston Crum.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Professor R. M. Foster, head of Department of Mathematics of Polytechnic Institute of Brooklyn, represented the Association at the inauguration of President J. H. Davis of Stevens Institute of Technology on October 12, 1951.

Assistant Professor V. L. Klee, Jr., of the University of Virginia has been appointed National Research Fellow and is spending the year at the Institute for Advanced Study.

The following have been appointed to positions at the Armed Forces Security Agency, Washington, D. C.: Mr. Joseph Blum, previously mathematician at the National Bureau of Standards, as analyst; Mr. F. T. Dresser of the University of Virginia as analyst; and Associate Professor W. R. Murray, Wagner College, as mathematician.

Dr. C. M. Ablow of Brown University, Associate Professor R. E. Gaskell of Iowa State College of Agriculture and Mechanic Arts, and Assistant Professor W. M. Stone, Oregon State College, have accepted positions at Boeing

Airplane Company, Seattle, Washington.

The following appointments have been made at the Los Alamos Scientific Laboratory, New Mexico: Assistant Professor O. W. Rechard of Ohio State University and Associate Professor Andrew Sobczyk, who is on leave of absence from Boston University, have been appointed Staff Members; Graduate Student I. J. Cherry of Oregon State College, Mr. Stewart Schlesinger, formerly a student at Illinois Institute of Technology, and Mr. F. J. Wall, previously a mathematical analyst, have been appointed to research assistantships.

At Northrop Aircraft, Incorporated, Hawthorne, California: Mr. G. W. Fairchild, previously a graduate student at the University of California at Los Angeles, has been appointed an electronics laboratory analyst; Dr. G. E. Gourrich has been appointed Research Engineer.

The Woodward Governor Company, Rockford, Illinois announces: Miss Beverly Marshall of the University of Illinois and Mr. Howard Capwell have been appointed Mathematicians; Dr. Rufus Oldenburger is serving as Chief Mathematician.

Mr. W. R. Allen of the Undergraduate Division of the University of Illinois has a position as Mathematician at the University of Chicago.

Assistant Professor R. D. Anderson of the University of Pennsylvania is a member of the Institute for Advanced Study for the year 1951–52.

Graduate Assistant Jean M. Baldwin of the University of Oklahoma has been appointed to an instructorship at Lamar State College of Technology.

Associate Professor D. H. Ballou of Middlebury College is on leave of absence for the year 1951–52 and has received an appointment as Visiting Associate Professor at Yale University in the program of Internships in General Education sponsored by the Carnegie Foundation.

Assistant Professor Joshua Barlaz of Rutgers University has been promoted to an associate professorship.

Mr. Alexander Basil, formerly a student at Hofstra College, is now an engineer at the Airborne Instruments Laboratories, Mineola, New York.

Mr. R. W. Beals, Jr., of Alfred University is teaching at Cattaraugus Central School, New York.

Associate Professor R. F. Bell of Eastern Washington College of Education has been appointed Head of the Department of Mathematics and Physics.

Mr. K. S. Bergman, previously a student at Gonzaga University, has been appointed a field assistant with the United States Geological Survey, Spokane, Washington.

Dr. D. W. Blackett, graduate assistant at Princeton University, has been promoted to the position of Research Associate.

Mr. H. F. Blasch, formerly a student at Hofstra College, is now in military service.

Instructor G. M. Bloom of Northwestern University has been appointed to an assistant professorship at Miami University.

Miss Dorothy C. Breynaert, who has been a student at the University of

New Hampshire, is now an actuarial student at Monarch Life Insurance Company, Springfield, Massachusetts.

Graduate Student E. B. Bridgforth of the University of Chicago has been appointed Statistician in the Nutrition Division of the Medical School, Vanderbilt University.

Assistant Professor N. A. Brigham of the University of Maryland has been appointed Mathematician in the Applied Physics Laboratory, Johns Hopkins University.

Mr. H. H. Brown, who has been a research engineer at Franklin Institute Laboratories, Philadelphia, Pennsylvania, has accepted a position at the Armour Research Foundation of Illinois Institute of Technology.

Mr. R. G. Brown, formerly research engineer at the Willow Run Research Center, University of Michigan, is now Project Engineer.

Graduate Student M. O. Burrell of Emory University is now a civilian trainer in the Department of the Air Force, Warner Robins Air Force Base, Georgia.

Professor C. H. Butler of Western Michigan College of Education has been appointed Head of the Department of Mathematics.

Mr. R. K. Butz has been appointed to an instructorship at the University of Georgia.

Assistant Professor E. L. Canfield of Drake University has been promoted to an associate professorship.

Associate Professor W. R. Cashen of the University of Alaska has been promoted to a professorship.

Dr. G. Y. Cherlin, who has been an instructor at Rutgers University, is now an actuarial student at Mutual Benefit Life Insurance Company, Newark, New Jersey.

Assistant Professor Nathaniel Coburn of the University of Michigan has been promoted to an associate professorship.

Assistant Professor Esther Comegys of the University of Maine has been promoted to an associate professorship.

Mr. W. V. Cressy, formerly a student at the University of California at Los Angeles, has accepted a position at Douglas Aircraft Company, El Segundo, California.

Assistant Professor A. W. Davis of Iowa State College has been promoted to an associate professorship.

Miss Anne F. Downey, previously a student at Regis College, has a position as Rate Specialist with the Liberty Mutual Insurance Company, Boston, Massachusetts.

Professor W. I. Dykes, head of the Department of Mathematics of South Texas Junior College, has succeeded the Reverend G. D. Pickens as Dean of the College.

Mr. R. L. Ely, who has been a research engineer with the Pittsburgh-Des Moines Company, is now in military service.

Teaching Fellow J. L. Ericksen of Oregon State College has received an appointment as Mathematician at the Naval Research Laboratory, Washington, D. C.

Professor G. M. Ewing is on leave of absence from the University of Missouri and is serving as Mathematician at Sandia Corporation, Albuquerque, New Mexico.

Assistant Professor A. B. Farnell is on leave of absence from the University of Colorado and has returned to military service.

Mr. Daniel Finkel has a position as Junior Actuary, New York State Insurance Fund, New York, New York.

Mr. E. L. Friedman, formerly a student at Massachusetts Institute of Technology, has accepted a position at the Chandler-Evans Division of the Niles-Bement-Pond Company, West Hartford, Connecticut.

Mr. A. L. Gilmore, Jr., has been appointed Instructor of Mathematics and Physics at Pearl River Junior College, Poplarville, Mississippi.

Mr. R. D. Glauz has been appointed to a research assistantship at Brown University.

Assistant Professor H. E. Goheen of Syracuse University has been appointed to an assistant professorship in the Department of Electrical Engineering of the University of Pennsylvania.

Mr. Isidore Goldman, formerly a mechanical engineer in the United States Naval Shipyard, has accepted a position as Design Engineer at Greer Hydraulics, Incorporated, Brooklyn, New York.

Assistant Professor N. A. Goldsmith of Illinois Wesleyan University has been appointed Head of the Department of Mathematics of Henderson State Teachers College, Arkansas.

Mr. N. E. Golovin has been appointed Assistant Director for Administration of the National Bureau of Standards, Washington, D. C.

Assistant Professor D. B. Goodner of Florida State University has been promoted to an associate professorship.

Mr. D. A. Gorsline, who has been teaching at Cambridge Central School, New York, has accepted a position with the Equitable Life Assurance Society, Albany, New York.

Instructor L. C. Graue of Sacramento State College has been promoted to a professorship.

Assistant Professor R. E. Graves of the University of Minnesota has a position at White Sands Proving Ground, Las Cruces, New Mexico.

Mr. F. D. Grogan, who has been teaching at Minden High School, Louisiana, has accepted a position as Chemical Corps Inspector, Dallas Chemical Procurement District, Texas.

Dr. Emil Grosswald, formerly a special lecturer at the University of Saskatchewan, is now a member of the Institute for Advanced Study.

Assistant Professor S. W. Hahn of Wittenberg College has been appointed to a professorship at Winthrop College.

Associate Professor P. R. Halmos is on leave of absence from the University of Chicago and is at Instituto de Matematica y Estadistics, Montevideo, Uruguay.

Mr. R. C. Haseltine of Swarthmore Center, Pennsylvania State College, has accepted a position as Assistant Research Engineer with the Burroughs Adding Machine Company, Philadelphia, Pennsylvania.

Graduate Assistant Melvin Henriksen of the University of Wisconsin has been appointed to an assistant professorship at the University of Alabama.

Mr. D. B. Hoagland of Trenton Junior College has been appointed Correspondence Instructor at the University of Missouri.

Dr. A. J. Hoffman, who has been a member of the Institute for Advanced Study, is now at the National Bureau of Standards, Washington, D. C.

Instructor R. E. Horton of Los Angeles City College has been recalled to active duty in the United States Air Force.

Mr. J. F. Hudson, previously a graduate student at the University of Tennessee, has a position as Statistical Assistant at the Carbide and Chemical Corporation, Oak Ridge, Tennessee.

Professor G. B. Huff of the University of Georgia gave a series of lectures entitled "a^n" at a colloquium of the University of Kentucky on July 31, August 1–2, 1951.

Miss Grace M. Hyder has accepted a position as Senior Actuarial Clerk for the Commonwealth of Massachusetts, Boston, Massachusetts.

Mr. M. A. Hyman, formerly a mathematician at the Naval Ordnance Laboratory, Washington, D. C., has received an appointment as Fulbright Scholar for the year 1951–52 and is located at Technical University, Delft, Netherlands.

Assistant Professor S. J. Jasper of Kent State University has been appointed to an associate professorship at East Tennessee State College.

Dr. Fritz John, who has been a mathematician at the Institute for Numerical Analysis, has been appointed to a professorship at New York University.

Assistant Professor G. K. Kalisch of the University of Minnesota has been promoted to an associate professorship.

Mr. Sidney Kaplan, a statistician for the Bureau of Employment Security, United States Department of Labor and a lecturer at Catholic University of America, has accepted a position as a research mathematician in the Office of the Comptroller of the Army, Washington, D. C.

Mr. Raymond Kassler, formerly a student at Brooklyn College, has been appointed mathematician at Evans Signal Laboratory, Belmar, New Jersey.

Mr. C. E. Kelley, who has been teaching in the Texas Public Schools, has accepted a position with the New York Life Insurance Company, New York, New York.

Lecturer E. H. Kingsley of Roosevelt College has been appointed mathematician in the Aerial Measurements Laboratory, Technical Institute, Northwestern University.

Mr. R. J. Kohlmeyer of Pratt Institute has been appointed to an instructorship at Lafayette College.

Instructor F. A. Kros of New Mexico College of Agriculture and Mechanic Arts is now a graduate student at the University of Colorado.

Mr. Charles Kurland, formerly a research engineer at Locomotive Development Company, Dunkirk, New York, has accepted a position as Research Engineer, Bell Aircraft Corporation, Niagara Falls, New York.

Graduate Student P. J. Leonard of Boston College has a position as Mathematician in the Applied Mechanics Branch, Watertown Arsenal, Massachusetts.

Research Assistant S. L. Levy of Brown University has been promoted to a research associateship.

Instructor H. D. Lipsich of the University of Cincinnati has been promoted to an assistant professorship.

Assistant Professor B. J. Lockhart of the United States Naval Postgraduate School has been promoted to an associate professorship.

Mr. A. L. Mayerson, previously an actuarial assistant at the Institute for Life Insurance, New York City, has accepted a position as Principal Actuary for the New York State Insurance Department.

Assistant Professor J. J. McCarthy of Rutgers University has a position as Project Engineer at Sperry Gyroscope Company, Great Neck, New York.

Miss Mary L. McLaughlin, formerly a student at Regis College, is now a graduate student at Boston University.

Assistant Professor F. A. McMahon of Manhattan College has accepted a position with the Sperry Gyroscope Company, Lake Success, New York.

Mr. J. O. Neilson, formerly a student at Augustana College, is now in the United States Air Force.

Associate Professor Albert Newhouse of the University of Houston has been promoted to a professorship.

Mrs. Theresa M. Oehmke has accepted a position as a computer in the Air Weapons Research Center of the University of Chicago.

Instructor Daniel Orloff of Southern Illinois University has been promoted to an assistant professorship.

Consulting Physicist N. G. Parke, III, has been named President of Parke Mathematical Laboratories, Incorporated, Concord, Massachusetts.

Mr. J. M. Patterson of New York University has accepted a position as Mathematician at Bendix Aviation Corporation, Detroit, Michigan.

Dr. W. D. Peoples, Jr., formerly a graduate assistant at the University of Georgia, has been appointed to an assistant professorship at Howard College.

Instructor K. S. Phelan of St. John's University has been promoted to an assistant professorship.

Assistant Professor J. W. Ponds of West Virginia State College is now a mathematician in the United States Air Force.

Professor C. A. Reagan of Friends University has retired with the title of Professor Emeritus.

Instructor C. L. Riggs of Kent State University has been appointed to an assistant professorship at East Texas State Teachers College.

Associate Professor Helen G. Russell of Wellesley College has been promoted to a professorship.

Mr. W. T. Sharp has a position as Assistant Research Officer of the National Research Council of Canada.

Mr. R. L. Shively, previously a teaching fellow at the University of Michigan, has been appointed to an instructorship at Western Reserve University.

Associate Professor F. C. Smith of the College of St. Thomas has been promoted to a professorship.

Mr. W. I. Steinkamp, formerly a student at Gonzaga University, is now in military service.

Reverend R. J. Swords of Weston College is a student at Rathfornham Castle, Dublin, Ireland.

Dr. D. L. Thomsen, Jr., research fellow at the California Institute of Technology, has been appointed a research engineer at the Jet Propulsion Laboratory, California Institute of Technology.

Assistant Professor C. W. Topp has a year's leave of absence from Fenn College and is a research associate at Case Institute of Technology.

Associate Professor H. L. Turrittin has returned to the University of Minnesota after spending a sabbatical year on the staff of Princeton University.

Professor Jacob Wolfowitz of Columbia University has been appointed to a professorship at Cornell University.

Instructor L. J. Zimmerman of Goshen College has been promoted to an assistant professorship.

Professor Emeritus W. T. Burns of Xavier University died on July 23, 1951

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THIRTY-SECOND SUMMER MEETING OF THE ASSOCIATION

The thirty-second summer meeting of the Mathematical Association of America was held at the University of Minnesota, Minneapolis, Minnesota on Monday and Tuesday, September 3–4, 1951, in conjunction with the summer meetings of the American Mathematical Society, the Institute of Mathematical Statistics, the Econometric Society, and Section A of the American Association for the Advancement of Science. A total of six hundred and twenty-three persons were registered, including the following two hundred and sixty-nine members

of the Association:

A. A. Albert, E. W. Anderson, R. D. Anderson, H. A. Antosiewicz, K. J. Arnold, C. F. Barr, M. A. Basoco, W. D. Baten, Helen P. Beard, E. G. Begle, J. H. Bell, Theodore Bennett, Brother Bernard Alfred, R. H. Bing, H. D. Block, K. H. Bracewell, J. W. Bradshaw, R. W. Brink, Charles Brumfield, R. C. Buck, L. J. Burton, L. E. Bush, J. H. Bushey, Jewell H. Bushey, W. H. Bussey, E. A. Cameron, R. H. Cameron, E. J. Camp, C. S. Carlson, Elizabeth Carlson, R. E. Carr, Abraham Charnes, Harold Chatland, S. S. Chern, E. W. Chittenden, A. G. Clark, F. Marion Clarke, H. D. Colson, C. H. Cook, T. F. Cope, A. H. Copeland, H. M. Cox, W. F. Crum, J. F. Daly, Marian E. Daniells, P. H. Daus, B. K. Dickerson, Flora Dinkines, M. H. Dipert, P. S. Dwyer, J. M. Earl, W. F. Eberlein, P. D. Edwards, H. P. Evans, F. D. Faulkner, H. E. Fettis, A. M. Feyerherm, C. H. Fischer, I. C. Fischer, Walter Fleming, E. E. Floyd, R. S. Fouch, H. D. Friedman, J. S. Frame, Abraham Franck, W. B. Fulks, W. R. Fuller, A. E. Gault, H. M. Gehman, Landis Gephart, C. B. Germain, Gladys Gibbens, W. M. Gilbert, R. E. Gilman, J. W. Givens, V. D. Gokhale, Michael Goldberg, D. B. Goodner, S. H. Gould, Arthur Grad, L. M. Graves, L. J. Green, V. G. Grove, Edwin Halfar, Mary E. Haller, P. C. Hammer, Frank Harary, W. L. Hart, Charles Hatfield, Charles Hatfield, Jr., C. E. Heilman, Ruth E. Henning, I. N. Herstein, M. R. Hestenes, A. J. Hoffman, R. V. Hogg, F. E. Hohn, D. L. Holl, Carl Holtom, E. Marie Hove, H. M. Hughes, Ralph Hull, M. Gweneth Humphreys, W. A. Hurwitz, W. H. Ito, C. G. Jaeger, G. H. Jaeger, D. A. Johnson, L. W. Johnson, E. R. Johnston, B. W. Jones, W. C. Kalinowski, G. K. Kalisch, Irving Kaplansky, Samuel Karlin, J. L. Kelley, L. M. Kells, D. E. Kibbey, E. C. Kiefer, V. L. Klee, Jr., Fulton Koehler, Max Kramer, W. H. Kruskal, O. E. Lancaster, R. E. Langer, E. H. Larguier, Rose Lariviere, L. S. Laws, W. G. Leavitt, J. R. Lee, D. H. Lehmer, F. C. Leone, C. H. Lindahl, Lee Lorch, W. S. Loud, C. C. MacDuffee, Saunders MacLane, H. M. MacNeille, Morris Marden, Margaret P. Martin, Ella Marth, Kenneth May, J. R. Mayor, Dorothy McCoy, S. W. McCuskey, W. C. McDaniel, W. R. McEwen, R. B. McHugh, J. V. McKelvey, Martha M. McKelvey, J. C. C. McKinsey, E. J. McShane, L. E. Mehlenbacher, Paul Meier, B. E. Meserve, D. M. Mesner, E. J. Mickle, J. J. Miller, W. H. Mills, Deane Montgomery, F. C. Mosteller, H. T. Muhly, Sigurd Mundhjeld, O. M. Nikodym, M. L. Norden, M. A. Nordgaard, E. A. Nordhaus, M. J. Norris, Rufus Oldenburger, Arthur Ollivier, J. M. H. Olmsted, Emma J. Olson, T. G. Ostrom, Margaret Owchar, O. J. Peterson, B. J. Pettis, H. P. Pettit, C. G. Phipps, George Piranian, A. E. Pitcher, J. C. Polley, George Polya, J. E. Powell, G. B. Price, G. C. Priester, F. M. Pulliam, A. L. Putnam, Tibor Rado, Henry Rainbow, J. F. Randolph, Ruth B. Rasmusen, L. T. Ratner, G. E. Raynor, M. O. Reade, W. T. Reid, Haim Reingold, H. B. Ribeiro, L. A. Ringenberg, D. D. Rippe, E. K. Ritter, J. H. Roberts, Fred Robertson, H. A. Robinson, P. C. Rosenbloom, Arthur Rosenthal, A. E. Ross, E. H. Rothe, Evelyn C. Rusk, Hans Samelson, A. C. Schaeffer, Henry Scheffe, Edith R. Schneckenburger, Nathan Schwid, R. R. Seeber, Jr., E. B. Shanks, C. R. Sherer, L. W. Sheridan, Harold Shniad, Sister M. Mercedes, Sister M. Prudentia, Sister M. Seraphim, Sister M. Teresine, M. F. Smiley, A. J. Smith, F. C. Smith, Ernst Snapper, W. S. Snyder, Andrew Sobczyk, Elizabeth Sokolnikoff, E. J. Specht, Abraham Spitzbart, R. C. Staley, W. L. Stamey, O. E. Stanaitis, H. E. Stelson, R. C. Stephens, M. H. Stone, Irwin Stoner, E. B. Stouffer, J. V. Talacko, P. Y. Tani, H. P. Thielman, G. H. M. Thomas, J. M. Thomas, Marian W. Thornton, Ella Thorp, C. B. Tompkins, Leonard Tornheim, J. I. Tracey, A. W. Tucker, J. W. Tukey, H. L. Turrittin, E. P. Vance, V. J. Varineau, E. C. Varnum, Bernard Vinograde, John von Neumann, J. A. Ward, S. E. Warschawski, K. W. Wegner, W. J. Wells, R. L. Westhafer, Louise A. Wolf, G. N. Wollan, R. S. Wollan, Y. K. Wong, J. W. T. Youngs, R. A. Zemlin, Antoni Zygmund.

Sessions of the Association were held on Monday afternoon and Tuesday morning in Room 150, Physics Building of the University of Minnesota. Vice-President L. M. Graves presided at the first session and President Saunders MacLane at the second session. The Program Committee for the meeting consisted of M. A. Basoco, Chairman, R. C. F. Bartels, and L. E. Bush.

FIRST SESSION OF THE ASSOCIATION

General Topic: "The Teaching of Undergraduate Collegiate Mathematics."

Address: "Let Us Teach Guessing," by Professor George Polya, Stanford University.

Panel discussion on the content of the undergraduate mathematical curriculum. Members of the panel: Professor M. R. Hestenes, University of California (Los Angeles) and National Bureau of Standards; Professor B. W. Jones, University of Colorado; Professor G. K. Kalisch, University of Minnesota; and Professor A. L. Putnam, University of Chicago.

Discussion and questions from the floor.

SECOND SESSION OF THE ASSOCIATION

"The Seven Functions of Ramanujan," by Dr. D. H. Lehmer, National Bureau of Standards.

"Applications of the Concept of Fairness of a Game to the Theory of Probability," by Professor J. L. Doob, University of Illinois.

Retiring Presidential Address: "The Things I Should Have Done, I Did Not Do," by Professor R. E. Langer, University of Wisconsin.

MEETING OF THE BOARD OF GOVERNORS

The Board of Governors of the Association met on Monday evening in the Conference Room of Pioneer Hall. Nineteen members of the Board were present. Among the more important items of business transacted were the following:

The resignation of E. N. Oberg as a member of the Board of Governors from the Iowa Section was accepted and W. M. Davis of Cornell College was elected to serve the balance of Professor Oberg's term. Professor W. B. Carver was reelected member of the Finance Committee for the four-year term, 1952–1955.

It was voted that the bequest to the Association by the late Professor Otto Dunkel be used to establish the "Dunkel Fund" of the Association. A committee was authorized to proceed with the publication of a book of problems from the MONTHLY to appear as a supplement to the MONTHLY. The cost of this supplement is to be charged against the Dunkel Fund. It was deemed appropriate that the book of problems be published as a memorial to Professor Dunkel because of his long association with the Problems Section of the MONTHLY.

It was voted to hold the 1952 summer meeting at Michigan State College, East Lansing, Michigan on September 1–2, 1952.

The following resolution was adopted:

"RESOLVED: The Board of Governors of the Association affirms its steady intention to conduct the scientific meetings, social gatherings, and other affairs of the Association so as to promote the interests of Mathematics without discrimination as to race, creed, or color. The President of the Association is hereby authorized and requested to determine the best means for avoiding dis-

crimination, by consultation on this subject with the various chairmen and secretaries of sections and other appropriate members of the Association, and to report the results of this consultation to the Board."

MEETING OF SECTION OFFICERS

A meeting of officers of the Sections of the Association was held on Tuesday evening in the Conference Room of Pioneer Hall. Representatives were present from twenty-two of the twenty-five Sections of the Association.

Professor H. A. Robinson described the programs of the Southeastern Section of the Association. Several other section officers described the programs of their respective sections. Comments were made regarding reports of section meetings and arrangements for scheduling meetings of the sections.

The relationship of a sectional governor to his section was discussed. It was the consensus of the group that if a sectional governor moves permanently from the geographical area of the section from which he was elected, it would be best for him to resign as a sectional governor.

Special activities of several sections were discussed. Brother Bernard Alfred reported on the contests for high school students conducted by the Metropolitan New York Section. Professor R. L. Westhafer described the plan of the Southwestern Section for sponsoring a traveling lecturer.

MEETINGS OF OTHER ORGANIZATIONS

The sessions of the American Mathematical Society began on the afternoon of Tuesday, September 4, and continued through Friday afternoon, September 7. The colloquium lectures on "Topological Transformation Groups" were delivered by Professor Deane Montgomery of the Institute for Advanced Study. Invited addresses were given by Professor R. H. Bing of the University of Wisconsin on "Partitioning Continuous Curves" and by Professor G. W. Whitehead of the Massachusetts Institute of Technology on "Homotopy Theory."

The Institute of Mathematical Statistics and the Econometric Society met jointly and separately beginning on Tuesday morning and continuing through Friday afternoon. At a joint session on Thursday afternoon, Professor Harold Hotelling of the University of North Carolina delivered the Rietz Memorial Lecture on the subject: "The Behavior of Standard Statistical Tests under Non-Standard Conditions." Three joint sessions were devoted to a "Symposium on Games, Decision Problems and Related Topics."

Section A of the American Association for the Advancement of Science held a single session on Wednesday morning at which Professor E. J. McShane of the University of Virginia delivered his retiring vice-presidential address on "Order Preserving Mappings of Partially Ordered Spaces."

ARRANGEMENTS, ENTERTAINMENT AND RECREATION

The Committee on Arrangements for the meeting consisted of J. M. H. Olmsted, Chairman, R. H. Cameron, H. M. Gehman, W. L. Hart, G. K.

Kalisch, Fulton Koehler, W. S. Loud, S. E. Warschawski, and J. W. T. Youngs.

Registration headquarters was in the lobby of Pioneer Hall. Rooms in Pioneer and Centennial Halls of the University of Minnesota were available to all attending the meetings and to their families from Sunday afternoon, September 2, until noon on Saturday, September 8. Meals were served in the cafeteria of Coffman Memorial Union.

On Tuesday evening an informal reception was held at the Campus Club under the sponsorship of the mathematical organizations and the College of St. Thomas. Arrangements were made for dancing, bridge and billiards during the evening.

On Wednesday evening a banquet for the mathematical organizations was held in the main ballroom of Coffman Memorial Union. Professor W. A. Hurwitz of Cornell University acted as toastmaster. President Saunders MacLane presented the greetings of the Association and spoke on the general topic of fashions in mathematics. President John von Neumann spoke on behalf of the American Mathematical Society. President P. S. Dwyer responded for the Institute of Mathematical Statistics and Professor Gerhard Tintner for the Econometric Society. Professor Ralph Hull of Purdue University presented a resolution expressing the thanks of the mathematical organizations to the officers of the University of Minnesota for making the facilities of the University available for the 1951 summer meeting.

On Thursday evening a recital of early keyboard music was given by Emilie Pray, concert pianist and instructor at Macalester College. The recital was held in Scott Hall on the University of Minnesota Campus.

In addition to the entertainment described above, special arrangements were made for the women attending the meetings. On Tuesday afternoon the Betty Crocker kitchens of the General Mills Corporation were visited. On Thursday a luncheon was held at the country club house of the Automobile Club of Minneapolis, overlooking the valley of the Minnesota River.

A particular debt of gratitude is due to the local members of the Committee on Arrangements for their untiring efforts to care for the visiting mathematicians.

H. M. GEHMAN, *Secretary-Treasurer*

MARCH MEETING OF THE MICHIGAN SECTION

The March meeting of the Michigan Section of the Mathematical Association of America was held on March 24, 1951, in the new Mathematics-Physics Building at Michigan State College in East Lansing in conjunction with the meetings of the Michigan Academy of Science, Arts, and Letters. Professor D. C. Morrow, Chairman of the Section, presided at both morning and afternoon meetings as well as at the business meeting held immediately following the luncheon in the Union. Professor J. H. Bell acted as local assistant chairman, helping the secretary make local arrangements.

The Michigan State College Mathematics Department served refreshments

in their seminar and meeting room, and their over-all activity as good hosts was recognized in a unanimously passed vote of thanks at the conclusion of the afternoon meeting.

A total of sixty-five persons attended the meeting including the following forty-eight members of the Association, N. H. Anning, J. W. Baldwin, J. M. Barbour, R. C. F. Bartels, W. D. Baten, F. A. Beeler, J. H. Bell, J. E. Bellardo, C. H. Butler, R. V. Churchill, C. J. Coe, P. C. Cox, J. W. Coy, C. C. Craig, D. A. Darling, D. E. Deal, P. W. Edmonson, Meta M. Ewing, V. G. Grove, H. H. Hannon, Gerald Harrison, Fritz Herzog, E. H. C. Hildebrandt, T. H. Hildebrandt, E. E. Ingalls, L. G. Johnson, P. S. Jones, Wilfred Kaplan, L. M. Kelly, H. D. Larsen, K. B. Leisenring, W. J. McClintock, L. E. Mehlenbacher, D. C. Morrow, H. W. Nace, A. L. Nelson, J. D. Novak, Mrs. Mary H. Payne, George Piranian, J. E. Powell, G. Y. Rainich, M. O. Reade, C. C. Richtmeyer, T. H. Southard, E. M. Steinbach, Leonard Tornheim, C. P. Wells, J. L. Wilson.

At the business meeting the reports of the Secretary and Treasurer were read and approved. The latter report included a payment of ten dollars to the CARE book fund for European university libraries, and a note that the Section had made a profit of approximately fifty dollars on sales of its college guidance pamphlet and charts. This profit did not include charges for the first printing of 2,000 copies which were distributed free.

The nominating committee, which consisted of Professors L. E. Mehlenbacher, V. G. Grove, J. W. Baldwin, reported the names of Professor N. H. Anning for Chairman and Professor P. S. Jones for Secretary-Treasurer for 1951–52. They were elected unanimously. It was announced that Professor J. S. Frame had been elected Governor for his region in the election conducted by the Secretary of the Association. The 1952 meeting will be held on April 12 at the University of Michigan in Ann Arbor, Michigan.

The following papers were presented at the morning and afternoon sessions. Due to illnesses and other difficulties, papers 12, 13 and 17 were presented by title.

1. *Variances of the difference between means when there are two missing values*, by Professor W. D. Baten, Michigan State College.

This paper demonstrated how to determine the variances between "treatment" means when there are two missing values in a randomized block design. Estimates of the missing values were obtained by minimizing the error sum of square. Variances of the various differences of two means were obtained by employing the variances of the individual plot values in such a way as to eliminate dependence between the terms of these differences. The variances were obtained between means containing estimates of the missing values and those that do not contain either one of these estimates, as well as the variances of the differences between two means when each contains an estimate of a missing plot value. These variances were compared with variances when there were no missing data.

2. *An 18th century approximation to the equally tempered scale*, by Professor J. M. Barbour, Music Department, Michigan State College.

Strähle's rough approximation to the powers of the 12th root of 2 featured the intersections of a transversal with 13 lines dropped from an external point of equally spaced points upon a line.

With a change in the angle of the transversal, this becomes an excellent approximation.

3. *Conformal mappings and Peano curves*, by Professor George Piranian, Dr. C. J. Titus, Professor G. S. Young, University of Michigan.

Salem and Zygmund [*Duke Math. J.* 12 (1945), 569–578] have proved that certain gap series $\sum a_k z^{n_k}$ with $\sum |a_k| < \infty$ map the unit circle into a Peano curve. The authors construct a class of functions whose Taylor series converge uniformly but not absolutely on the unit circle, and which map the unit circle onto the closure of any preassigned bounded domain.

4. *An abelian semi-group of transformations of Hausdorff type*, by Mr. George Brauer, student, University of Michigan, introduced by Professor George Piranian.

This paper dealt with the semi-group of Loeplitz matrices of the form $e^{-1}\nu\rho$ where ν is an arbitrary diagonal matrix and where the Euler matrix ρ, used by Hausdorff, has been replaced by the Cesaro matrix of order one. Regularity conditions and convergence fields of these matrices were discussed.

5. *A survey of the industrial subsidy of the study of applied mathematics*, by Professor T. H. Southard, Wayne University and the Industrial Mathematics Society.

In the spring of 1950 the Industrial Mathematics Society's Education Committee asked twenty-odd industrial concerns and technical government agencies for information on the kinds of courses in "applied mathematics" (including mathematics courses beyond the calculus as well as "refresher" calculus, and such courses as hydrodynamics, elasticity, etc.) which they were offering their employees on the premises or elsewhere. Information was also sought concerning any subsidies (in the form of time-off, money, tuition, etc.) the concerns offered their employees while attending these courses. This was a report on the 20 replies received.

6. *An inverse two dimensional problem in elasticity*, by Mr. L. G. Johnson, Research Laboratories Division, General Motors Corporation.

In determining the residual stresses in a plate, the usual procedure is to remove thin strips from the plate and then to determine the residual stresses in the strips. However, the strip stresses will be different from the original plate stresses, and in order to convert strip stresses into plate stresses, it becomes necessary to solve a boundary value problem involving the biharmonic partial differential equation. The author discussed this boundary value problem and a method of arriving at some satisfactory solutions.

7. *The order of approximation of real numbers by rationals*, by Professor Fritz Herzog, Michigan State College.

For real x and positive α, let $K(x:\alpha) = \liminf q^{\alpha} |x - p/q|$, taken over all rational $p/q \neq x$ with $q > 0$. For each x, there is a unique value γ such that $K(x:\alpha) = 0$ for $\alpha < \gamma$ and $K(x:\alpha) = \infty$ for $\alpha > \gamma$. This number $\gamma = \gamma(x)$ is called the order of approximation of x. If x is itself rational then $\gamma(x) = 1$. If x is irrational let b_n denote the nth partial quotient (denominator) and A_n/B_n the nth convergent of the simple continued fraction for x. Then

$$\gamma(x) = 2 + \limsup \frac{\log b_{n+1}}{\log B_n}$$

This formula enables us to study the function $y = \nu(x)$. From the set-theoretical viewpoint, the function is highly "pathological"; it assumes any given value of y, $2 \leqq y \leqq \infty$, on a set of values x which has the power of the continuum in any real interval. On the other hand, from the measure theoretical viewpoint, the function is extremely "normal"; in fact, $\gamma(x) = 2$ for almost all x.

8. *Normalized coordinate vectors as a shorthand for Descartes*, by Dr. K. B. Leisenring, University of Michigan.

The point calculus of Grassman as applied to geometry by Forder (*Calculus of Extension*, Cambridge, 1941) is, in part, in the case of two dimensions formally identical with the algebra of a three-dimensional vector space, using vector and scalar products. Metrical properties, however, require a considerable elaboration of concepts not so familiar. If we treat ordinary homogeneous coordinates for the plane as coordinate vectors we can obtain a neat handling of metrical properties simply by suitable normalization.

A point vector is normal if its third component is unity; a line vector is normal if the sum of the squares of the first two components is unity. The line through A, B is represented by the vector product $[AB]$; the point of intersection of p and q by $[\underline{p\ q}]$. The basic metric relations are: if A and B are normal then $\pm[A\ B]$ divided by the distance AB is normal; if p and q are normal then $[p\ q]/\sin(p\ q)$ is normal.

If A and B are normal, $P=(\lambda A+\mu B)/(\lambda+\mu)$ is normal and gives a barycentric scale on the line. A-B is not normalizable and represents the point at infinity on AB which gives a simple handling of parallelism. The vector $S=(0, 0, 1)$ representing the unnormalizable line at infinity is useful in handling perpendicularity. The perpendicular to AB through P has a vector proportional to $A-B-S\ (A\cdot P-B\cdot P)$, A, B, P being normal.

A number of metrical theorems were shown to be equivalent to identities in these normalized vectors and two fundamental theorems on conics were obtained.

9. *On a theorem of Borsuk*, by Professor L. M. Kelly, Michigan State College.

In 1932 Borsuk proved that any decomposition of a sphere of diameter one in n-dimensional euclidean space into n mutually exclusive sets, leaves at least one of the "pieces" of diameter one. This note offered an elementary proof of this result in E_3. The proof was based on a map coloring idea (probably not new) and does not generalize to E_N.

10. *What shall we do with mathematics?* by Professor E. H. C. Hildebrandt, Northwestern University.

In his invited one hour address, Professor Hildebrandt listed as some of the most serious problems in mathematics education, including the postponement of the beginning of elementary work, the cutting off at the top of the more advanced secondary school courses, policies of excessive regular chronological promotion, the lack of equipment and teaching aids in mathematics classrooms, inadequately varied teaching methods, inadequately trained teachers especially on the elementary level.

Professor Hildebrandt suggested that the Association can work to improve this situation by seeking and stimulating the better students through junior academies of science and contests, by improving the guidance work done with all students through publication of information about mathematics and its uses, by improving teaching through preparation of books, lectures, teaching aids, and by working for improved teacher training programs and requirements for the certification of teachers.

A lively discussion followed Professor Hildebrandt's talk.

11. *The solving systems of linear equations in college algebra*, by Professor Gerald Harrison, Wayne University.

It was proposed that in the course commonly called college algebra the method of elimination for solving systems of any number of linear equations in any number of unknowns be carried through without the intervention of determinants. The step-wise method of elimination can be easily justified to the student. He should be given the appropriate notation of "arrays" or "matrices" and shown how to systematically apply the elementary row operations to the array to ob-

tain the solution of any system of linear equations. This manner of solving system of linear equations has the advantages of unifying the subject, of being easily understandable to the student, of being a useful computational method in many cases superior to the determinantal method, and of convincing the student that a system of many linear equations in many unknowns is not in any essential respect more difficult to solve than a system to a few equations in a few unknowns.

12. *By scissors from n squares to one*, by Professor B. M. Stewart, Michigan State College.

For any given positive integer, there exists a positive integer k (for example, $k=8^n$), presumably dependent on n, such that if each of n unit squares is dissected in exactly the same way into k convex parts, then the kn pieces thus obtained may be composed, under rotation and translation, but excluding reflection, into one square of area n. The problem proposed, but still open, is that of deciding whether there is a k independent of n. The best result found is that $k=3$ will suffice if n is the sum of two squares of integers. Many other special results are obtained.

13. *A simple device for empirical sampling from an arbitrary distribution*, by Professor H. W. Alexander, Adrian College.

By the use of an appropriately constructed circular scale and a freely-spinning pointer, random samples may be drawn from an arbitrary population. This device may be used to illustrate the genesis of various well-known distributions derived from the normal distribution, and to study empirically various methods of industrial quality control.

14. *A problem of geometry suggested by G. B. Price*, by Professor Wilfred Kaplan, University of Michigan.

The following question was raised by Professor G. B. Price in a letter to Professor Kaplan: Is every homeomorphism of the disc $x^2+y^2<1$ into the plane which maps line segments onto line segments necessarily a collineation? This is answered affirmatively by the following reasoning. Let ABC be a triangle in the given disc and let D, E, F be the midpoints of AB, AC and BC. The given homeomorphism T can be followed by a collineation T_1 such that T_1T leaves A, B, C, D, E, fixed. Further, T_1T takes line segments to line segments. From these properties one deduces that T_1T must also leave F fixed, as well as the midpoints of AD, DB, AE, EC. By induction one now establishes that T_1T leaves fixed each point of a set dense on the periphery of ABC; hence, by continuity, T_1T leaves every point on the periphery fixed. From this one concludes that T_1T leaves all points of the disc fixed. Thus T is simply the inverse of T_1, is hence a collineation.

15. *On a method of averaging*, by Professor D. A. Darling, University of Michigan.

Some convexity arguments were used to define a general method of averaging, and to deduce conditions under which the averages can be ordered. Some of the classical inequalities appeared as special cases.

16. *Matrix equations*, by Professor J. H. Bell, Michigan State College.

This paper summarized the algorithm of M. H. Ingraham for the solution of the unilateral matrix equation together with a generalization to the case in which the coefficients are not square. The methods were then extended to solve a unilateral direct product equation and to indicate a method of solution of simultaneous unilateral equations.

17. *Configuration problems in distance spaces*, by Professor E. A. Nordhaus, Michigan State College.

Configurations having specified properties arising from point sets in metric or semi-metric

spaces were studied. A generalization of a theorem due to Erdös and Anning was given, and related questions and unsolved problems were discussed.

P. S. JONES, *Secretary*

APRIL MEETING OF THE OHIO SECTION

The thirty-fifth annual meeting of the Ohio Section of the Mathematical Association of America was held at the Ohio State University, Columbus, Ohio, on Saturday, April 21, 1951. Professor V. C. Stechschulte, Chairman of the Section presided at the morning and afternoon sessions.

One hundred seven persons registered attendance, including the following seventy-nine members of the Association: J. E. Adney, Jr., C. E. Amos, W. E. Anderson, P. R. Annear, D. F. Atkins, Grace M. Bareis, H. M. Beatty, William Beck, Theodore Bennett, Foster Brooks, C. D. Calhoon, Dorothy I. Carpenter, W. G. Clark, E. H. Clarke, W. F. Cornell, H. K. Crowder, Grace M. Cutler, Wayne Dancer, L. E. Davis, Jr., R. C. Davis, Violet B. Davis, R. E. Dowds, R. H. Downing, B. B. Dressler, P. L. Evans, H. E. Fettis, B. E. Gatewood, G. R. Glabe, E. L. Godfrey, S. W. Hahn, H. G. Harp, R. G. Helsel, I. N. Herstein, P. S. Herwitz, J. S. Hokanson, R. Y. Iwanchuk, S. J. Jasper, E. D. Jenkins, Margaret E. Jones, O. C. Juelich, John Kaiser, Chosaburo Kato, D. M. Krabill, D. H. Kraft, H. D. Lipsich, L. L. Lowenstein, R. H. Marquis, H. R. Mathias, C. C. Morris, O. M. Nikodym, Helen Olney, Emma J. Olson, S. R. Orr, C. F. Pinzka, H. S. Pollard, Tibor Rado, O. W. Rechard, Dorothy Renzema, R. F. Rinehart, D. T. Ross, S. A. Rowland, H. J. Ryser, K. C. Schraut, Samuel Selby, Sister M. Constantia, E. T. Stapleford, V. C. Stechschulte, Andrew Sterrett, Michael Tikson, H. E. Tinnappel, H. S. Toney, W. R. Transue, E. P. Vance, E. H. Wang, D. R. Whitney, R. B. Wildermuth, F. B. Wiley, C. O. Williamson, Alberta Wolfe.

The following officers were elected for the coming year: Chairman, R. F. Rinehart, Case Institute of Technology; Secretary-Treasurer, Foster Brooks, Kent State University; Member of the Executive Committee, L. H. Miller, The Ohio State University; Program Committee, H. R. Mathias, Bowling Green State University (Chairman), W. R. Transue, Kenyon College, L. L. Lowenstein, Kent State University.

The following papers were presented:

1. *Some mathematics in seismology*, by Professor V. C. Stechschulte, Xavier University.

Mathematical techniques of many kinds find application in various phases of seismology. Engineering seismology and instrumental seismology were mentioned as utilizing particularly the mathematics of vibrating systems. After a brief discussion of the various types of earthquake waves and of the character of the data supplied by seismograms, the contribution of statistical methods in seismology was described. Particular attention was given to the method wherein the solution of an integral equation (essentially of the kind involved in Abel's problem) leads to the determination of the velocity of earthquake waves at various depths within the earth. Some geometrical methods by which data from deep-focus earthquakes were made particularly useful for this velocity-depth problem were discussed.

2. *Canonical forms for matrices with elements in a finite field*, by Mr. Glenn Stahly, The Ohio State University, introduced by Professor D. R. Whitney.

An outline of the main results obtained in the study of matrices with elements in various fields was given. In particular, some results of L. E. Dickson's concerning congruence of matrices with elements in finite fields were put in more modern mathematical language. First, some definitions were given, after which congruence was discussed for matrices in various fields. Then number theory was considered, for some of these results were needed in succeeding work. Finally, canonical forms for matrices with elements in finite fields were obtained.

3. *Simple difference sets* (mod v) *for v less than* 2,560,000, by Mr. T. A. Evans and Professor H. B. Mann, The Ohio State University, introduced by the Chairman.

A set $a_1, a_2, \cdots, a_k$ of different residues (mod v) is called a difference set (v, k, λ) if the congruence $a_i - a_j \equiv d \pmod{v}$ has exactly λ solutions for every $d \not\equiv 0 \pmod{v}$. Necessary and sufficient conditions for the existence of a difference set (v, k, λ) are not known. However Singer has demonstrated the existence of simple difference set $(v, k, 1)$ if $k-1=n$ is a prime power and $v=n^2+n+1$. The conjecture has been made that this is a necessary condition as well as a sufficient one for the existence of a difference set $(v, k, 1)$. By means of diverse theorems, this speaker has shown that the only simple difference sets existing for $v \leqq 2{,}560{,}000$ are those for which $k-1=n$ is a prime power and $v=n^2+n+1$.

4. *The ubiquitous golden section*, by Professor Wayne Dancer, University of Toledo.

The author pointed out the diverse places in the field of mathematics where the idea of the golden section occurs. This ubiquitous pattern is encountered not only in various kinds of geometry, but in arithmetic, algebra, trigonometry, and analysis as well. Sequences of integers first found in ancient Greek "ladders," and later known as the Fibonacci sequence, furnish approximations to the ratio of the segments involved in the golden section. These approximations are identical with the convergents of the continued fraction representing this ratio, that is, the positive root of the quadratic $x^2+x=1$. Both positive and negative powers of x reduce to binomials of the form $a+bx$, where a and b are successive elements of the sequence mentioned above. A rectangle or an isoceles triangle with sides in the ratio $x:1$ can be subdivided into similar rectangles and triangles whose vertices lie on a logarithmic spiral.

5. *The interactions of modern mathematics and modern physics*, by Professor Alfred Schild, Carnegie Institute of Technology.

This was an invited address. The interaction of mathematics and physics has provided valuable stimuli in both fields. This is illustrated in particular by tracing the central role played by one concept, namely invariance, in the twentieth century development of special relativity, of the theories of the Dirac electron and of elementary particles, of general relativity, and of non-Riemannian differential geometry.

This development has been limited by the small number of mathematicians involved. This is due not to a lack of interest, but rather to an ignorance arising from lack of opportunity. It is suggested that students of mathematics be encouraged to become familiar with the simple basic concepts of modern theoretical physics.

6. *Bounded variation for functions of two variables*, by Professor W. R. Transue, Kenyon College.

It was the purpose of this paper to describe the definitions of bounded variation for functions

of two variables given by Vitali and Frechet, and to discuss the properties of functions which have bounded variation according to these two definitions. A comparison was made of the two definitions, and examples were given to clarify their meanings. For functions of bounded Frechet variation, two-dimensional analogues of the following properties of functions of bounded variation of a single variable were presented: (a) The existence of a limit as a point is approached from the left or from the right; (b) Countability of the points of discontinuity of a function of bounded variation. These analogues are: (a) The existence of a limit as a point is approached from any one of the four open quadrants defined by lines through the point parallel to the coordinate axes; (b) Covering the points of discontinuity by a countable number of lines parallel to the coordinate axes.

7. *Preparation for graduate work in mathematics*, an informal discussion led by Professor D. R. Whitney, The Ohio State University.

8. *Classroom Notes*. Professor John Kaiser, Kent State University, presented a simple method for motivating the definition of e^{iz}.

FOSTER BROOKS, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-fifth Annual Meeting, Brown University, Providence, Rhode Island, December 29, 1951.

Thirty-third Summer Meeting, Michigan State College, East Lansing, Michigan, September 1–2, 1952.

The following is a list of the Sections of the Association with dates of future meetings as far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Waynesburg College, Waynesburg, Pennsylvania, May 10, 1952.
ILLINOIS, Western Illinois State College, Macomb, May 9–10, 1952.
INDIANA, Indiana University, Bloomington, Spring, 1952.
IOWA, Coe College, Cedar Rapids, April 18–19, 1952.
KANSAS
KENTUCKY, University of Kentucky, Lexington.
LOUISIANA-MISSISSIPPI, Northwestern State College, Natchitoches, Louisiana, February 15–16, 1952.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, National Bureau of Standards, Washington, D. C., December 8, 1951.
METROPOLITAN NEW YORK, Spring, 1952.
MICHIGAN, University of Michigan, Ann Arbor, April 12, 1952.
MINNESOTA
MISSOURI, Lindenwood College, St. Charles, Spring, 1952.
NEBRASKA
NORTHERN CALIFORNIA, Stanford University, Stanford, January 26, 1952.
OHIO, April 19, 1952.
OKLAHOMA
PACIFIC NORTHWEST, University of Oregon, Eugene, June 20, 1952.
PHILADELPHIA, University of Pennsylvania, Philadelphia, November 24, 1951.
ROCKY MOUNTAIN, Western State College, Gunnison, Colorado, May, 1952.
SOUTHEASTERN, Georgia Institute of Technology and Agnes Scott College, Atlanta, March 21–22, 1952.
SOUTHERN CALIFORNIA, Occidental College, Los Angeles, March 8, 1952.
SOUTHWESTERN, University of Arizona, Tucson, April 11–12, 1952.
TEXAS, East Texas State Teachers College, Commerce, April, 1952.
UPPER NEW YORK STATE, Hobart and William Smith Colleges, Geneva, May, 1952.
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